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GROWTH CORRELATES OF ELECTROMOTIVE FORCES IN MAIZE SEEDS

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Each living maize seed being a physicochemical system has an inherent electrical potential which is measurable after the seed has been soaked for several hours in water. This potential is presumably the sum of the e.m.f.'s of each individual cell, and as such the degree of magnitude of this force may be expected to show correlations with growth processes which are then being initiated. Burr (1943) has already shown a connection between the mean potential of an inbred line and its probable heterotic value in crosses. He further showed the large potential difference implicit in two inbred lines whose large size differences are conditioned by a single gene, the recessive allele in a homozygous condition, resulting in diminution of plant and ear size. This seems to point rather clearly to a positive relationship between the electrical potential and the genetic constitution. Correlations between plant growth and the electrical potential are the easiest way to elucidate this relationship, and it was with this thought that this series of investigations was begun.

The technique employed was as follows. The seeds were placed in tap water for 18–24 hours before measurement. A seed was picked at random after soaking and placed on a cone of plasticene located in the center of a revolving stage (Fig. 1). Silver-silver chloride electrodes in physiological salt solution, terminating in camel's hair brushes, were mounted in a Zeiss manipulator and brought into contact with the micropylar and germinal ends of the seed. The electrodes were connected by flexible unshielded copper wire to the input binding posts of a d. c. microvoltmeter (Burr, 1936). The magnitude of the potential difference was read from a standard galvanometer calibrated to read in millivolts. By means of the revolving stage the seed was rotated 180° and a second determination made. This was repeated until consistent readings were obtained; usually this re-

quired 3 to 5 measurements. A large number of tests were made to determine the validity of the above procedure.

Certain consistencies are encountered in the measurement of the e.m.f. in maize seeds. Measurements can be taken in any one of the three dimensions with a measurement in the longitudinal axis giving the largest potential figure for any given seed. The measurements of e.m.f. from side to side and back to front are always smaller and generally proportional to that of the long axis with the measurement from side to side being usually larger than from back to front.

When e.m.f. measurements are taken on the long axis, it is found that the micropylar end is almost always negative to the germ end of the seed.

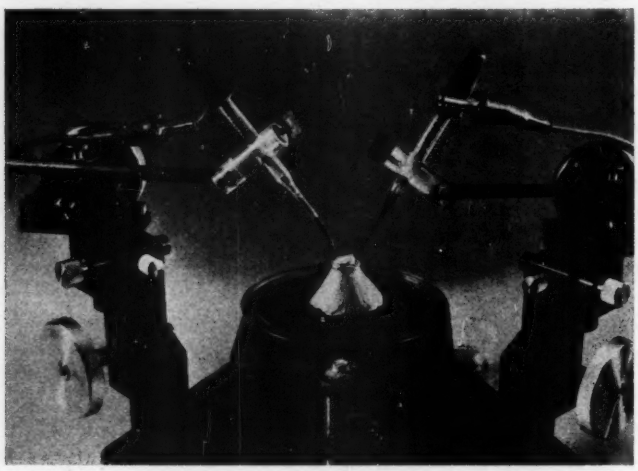

FIGURE 1

A photograph of the electrode placement on the maize seed.

Further, if an electrode is placed at the micropylar end, and the other electrode is shifted about the surface of the seed, it is found that there is a regular pattern of equipotential lines over the surface (Fig. 2).

Moreover, it is found that two e.m.f. measurements can be taken on any given seed after soaking, and that both of these measurements can be shown to be correlated with certain measurable attributes of later plant development. The first potential is the reading obtained when the electrodes first touch the seed. The galvanometer reading must be taken immediately for the needle falls off rapidly until within thirty to one hundred and twenty seconds it has attained a stable point at which point it will remain constant for two to five minutes. This falling off in potential and

subsequent attainment of a fixed potential is presumably due to an ionic equilibrium between the seed and the conducting solution of NaCl. This second potential is designated as the equilibrium potential, and is the reading on which Burr based his ideas of the correlation between electrical potential and genetic constitution. The first or unstable potential reading will be designated as the prime potential.

Prime Potential.—In the spring of 1945, a five-by-five latin square test was set up in which the five entries were taken from a commercial sample of double-crossed field corn hybrid, U. S. 13, on the seeds of which the prime potential was taken. The entries in the test were: (1) low prime potential, (2) medium prime potential, (3) untested seed, (4) high prime potential and (5) untested seed.

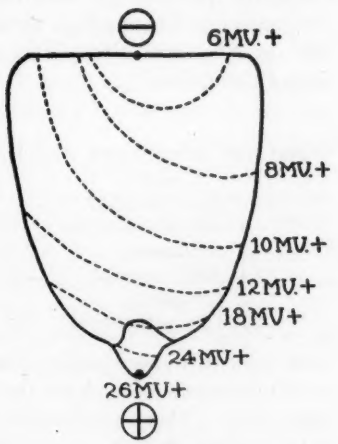

FIGURE 2

Equipotential lines on the surface of the maize seeds.

The seed of this test was planted directly in the field with a single seed being planted at each place. Analysis of the results shows that there was a highly significant difference in germination between entries (table 1).

TABLE 1

ANALYSIS OF VARIANCE FOR % GERMINATION (PRIME POTENTIAL TEST 1945)

SOURCE OF VARIATION	*DF*	*MS*	*F*
Rows	4	285.86	4.30*
Columns	4	67.06	1.01
Potential level	4	579.76	8.76†
Error	12	66.44	
Total	24		

* Significant on 0.05 level.
† Significant on 0.01 level.

Table 2 gives the percentage of germination for each entry. The high potential had a significantly higher percentage of germination than the low potential while the medium potential and the untested entries are much alike in percentage of germination and are also significantly higher than the low potential level. Table 2 also shows the yield figures in grams/100 for the five entries. Here again the high potential level and the two un-

tested entries are significantly higher than the low potential level. But the table also shows the obvious connection between the percentage of germination (and hence the plant number) and the yield for an entry. When the yield is corrected for discrepancies in stand by covariance (Singleton and Nelson 1945), these differences are all obviated showing that only the differential germination affected yield and nothing inherent in the entries themselves.

TABLE 2

GERMINATION PERCENTAGES AND YIELD IN GRAMS/100 FOR PRIME POTENTIAL LEVELS

ENTRY	% GERMINATION	YIELD
1 (Low potential)	63	211
2 (Medium potential)	79	242
3 (Untested)	83	269
4 (High potential)	92	277
5 (Untested)	85	271
J. S. D. on 0.01 level,	16	55

In addition to germination, the heights of the plants were measured at weekly intervals throughout the season, and silking and tasselling dates were taken. There was practically no difference in silking and tasselling dates between entries. At no time during the season were the heights of the various entries more than minutely different.

This evidence seems to point to the conclusion that the prime potential is a measurement indicative of seed quality, irrespective of genetic constitution. The seeds from the high potential level will germinate significantly better than those from the low potential level but an advantage is not apparent even in seedling vigor once the plants have emerged from the ground, and no variations other than differential germination can be shown between the entries.

Equilibrium Potential.—The stable measurement characteristic of a given seed after it has been in contact with the electrodes for thirty seconds up to two minutes is almost invariably smaller than the prime potential. Just how closely it approximates the prime potential varies from seed to seed and in a larger sense from variety to variety.

In 1943 equilibrium potentials were measured on nine inbred and hybrid lines of corn. The mean measurements and range for each line are given in table 3.

The comparative potentials of the inbred and hybrid lines are in agreement with Burr's findings that the potential of a hybrid between two inbreds is nearly always larger than the potential characteristic of the inbred with the smaller potential, and may be larger than the potential of either parent. Unfortunately, the inbred Ohio 40B which enters into two of the combinations was not tested or grown, and it is rather difficult to explain the excess of potential difference of 40B × 38-11 over 40B × L317 since

L317 itself has a significantly higher potential than 38-11. However, if the potential measurement is correlated with hybrid vigor, this would be explained since 40B and L317 are both Lancaster inbreds and, in general, would not be expected to show as much hybrid vigor when crossed as a cross of 40B × 38-11 which are two unrelated inbreds.

The readings of 18 for $(Wf \times Kr)F_2$ and 15 for $(Wf \times Kr)F_1$ agree with other comparisons of potentials of F_1 and F_2 seed from the same cross made here at the Connecticut Station. The F_2 seed has a slightly but not significantly higher potential.

Within any line (inbred or hybrid) there is approximately normal distribution of the potential readings about a mean. Seed of these nine entries was divided into high, low and medium potential within each entry. This division served as a basis for replication in a randomized block test since it was not believed that potential differences within entries would be significant. In the following results we have eliminated the medium division of each entry from consideration since retention of this division leads to a constant intergradation of potential reading for the entry where we desire two clearly defined groups.

TABLE 3

MEAN EQUILIBRIUM POTENTIAL IN MILLIVOLTS—1943

	X POTENTIAL	RANGE
Ind. 38-11, 1941 seed	17	9–26
Iowa L317, 1941 seed	27	7–52
38-11 × L317, 1942 seed	21	5–35
L317 × 38-11, 1941 seed	22	3–45
38-11 × L317, 1941 seed	32	15–50
Ohio 40B × 38-11, 1942 seed	39	12–60
Ohio 40B × L317, 1942 seed	28	11–44
$(Wf \times Kr)F_2$, 1941 seed	18	6–46
$(Wf \times Kr)F_1$, 1941 seed	15	2–35

In the field test there was no significant difference between high and low divisions of any entry for percentage of germination. The germination percentage for all highs was 87.8 while for all lows it was 83.2, the difference being 4.6. Measurements were taken of the height of plants at weekly intervals throughout the season and figures 3–5 show the plant height in inches for both high and low divisions of each entry plotted against the time in days. There is also a summation in the tenth graph with plant heights averaged for all highs and all lows plotted against time in days. It is evident that in seven of the nine entries, there was a noticeable tendency for the highs to grow faster and attain a greater height by the end of the growing season with the segregation of high and low growth rates occurring early in the season and exactly corresponding with the high and low potential reading. In the other two entries there was a segregation of high

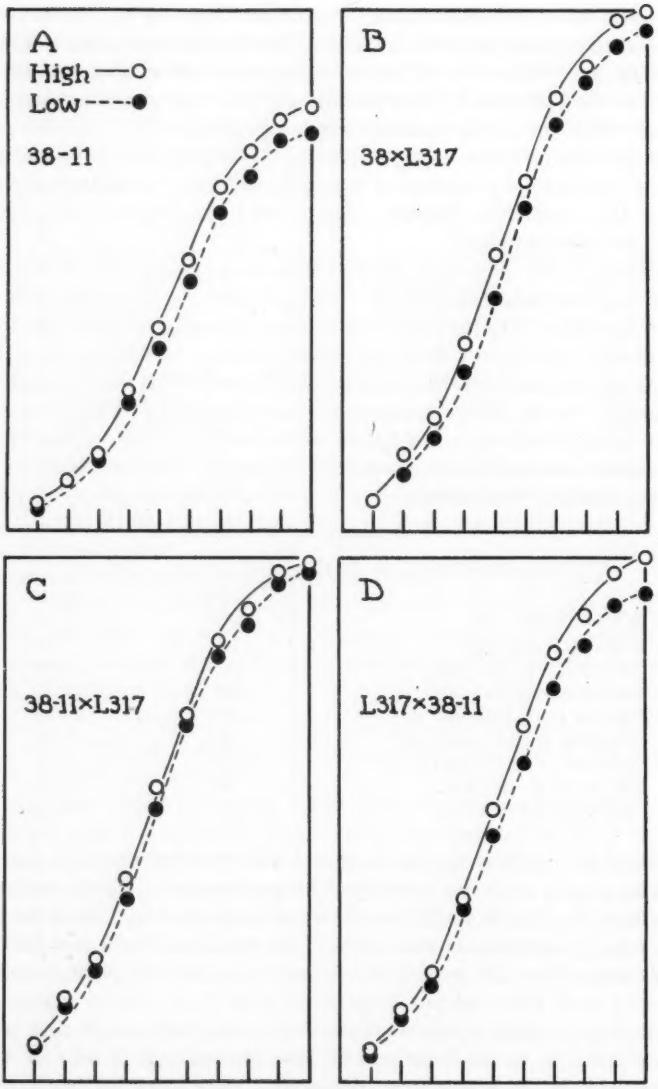

FIGURE 3

Graphs of the height of corn plants at weekly intervals from 4 measured strains.

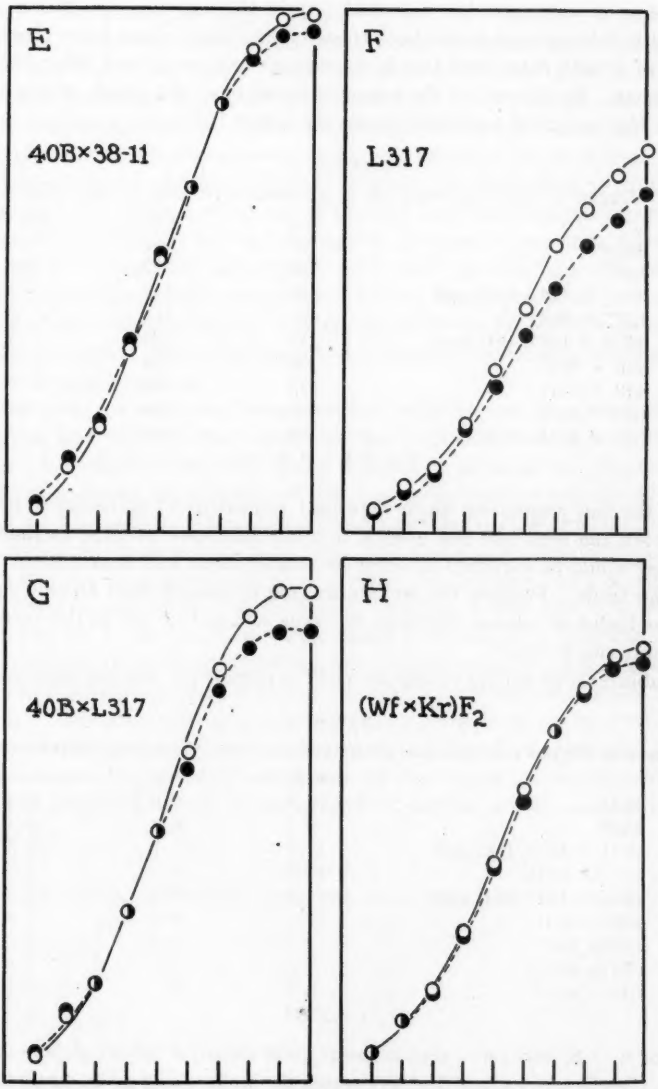

FIGURE 4

Graphs from 4 additional lines.

and low growth rates but not until late in the season. The noteworthy point in this instance is that both these entries, where there was no segregation of growth rates until late in the season, have an inbred, Ohio 40B, in common. By the end of the season in every case, the plants which came from high potential seeds had grown the tallest (table 4).

TABLE 4

HEIGHT IN INCHES OF HIGH AND LOW DIVISIONS OF EACH ENTRY—1943

	HIGH	LOW	HIGH-LOW
38-11	97	91	6
L317	88	77	11
38-11 × L317, 1942 seed	120	116	4
L317 × 38-11	120	112	8
38-11 × L317, 1941 seed	119	116	3
40B × 38-11	119	115	4
40B × L317	113	106	7
$(Wf \times Kr)F_2$	100	96	4
$(Wf \times Kr)F_1$	108	104	4

$t = 6.54$ $P < 0.001$

If the true population mean were zero, indicating no difference in height between the high and low groups, a mean difference as large as that observed would be expected to occur by chance alone less than once in 1000 similar trials. Further, the probability that by chance alone all nine values in the high-low column will be of the same sign as they are in this instance are only two in 512.†

Table 5 shows similar results for yield in pounds per plot for each entry.

TABLE 5

YIELD IN POUNDS PER PLOT FOR HIGH AND LOW DIVISION OF EACH ENTRY—1943

	HIGH	LOW	HIGH-LOW
38-11	1.4	0.6	0.8
L317	0.7	0.3	0.4
38-11 × L317, 1942 seed	11.4	6.1	5.3
L317 × 38-11	11.4	9.8	1.6
38-11 × L317, 1941 seed	11.5	8.3	3.2
40B × 38-11	11.0	9.4	1.6
40B × L317	9.0	10.3	−1.3
$(Wf \times Kr)F_2$	7.0	3.1	3.9
$(Wf \times Kr)F_1$	8.4	8.5	−0.1

$t = 2.429$

For $n = 8$, only one value in twenty will exceed 2.306 by chance alone. It is, therefore, apparent that separating the seeds of any given strain into a high and low group by their equilibrium potential has in the main been effective in placing the plants of superior growth characteristics in one category and the plants of inferior growth characteristics in the other category. Perhaps the e.m.f. measurement is a rough index of the physiological ef-

ficiency of a given genotype. In this connection it is rather unexpected that the divisions in inbred lines should as be divergent in growth as the L317 and 38-11 lines tested here. However, Jones (1945) has found unexpected variability in inbreds after many years of inbreeding and indications of heterotic vigor in sublines when crossed together. These appeared as morphological mutants in segregating and homozygous progenies. Presumably there are also subvisible mutants, the manifestations of which are limited to slowing the growth rate of the plant and lessening total growth.

In 1944, three lines were subdivided on the basis of potential difference and grown in separate latin square field tests for each line. Number 1 was a very uniform field corn inbred C20, the second was the F_2 of a cross of two field corn inbreds ($Wf \times Hy$), the third was the F_2 of a cross of a field corn inbred C243 $\times$ 1062 (a genetic tester for chromosome 1 and carrying br/br, bm_2/bm_2, f_1/f_1).

The seeds for each entry were weighed after the test for potential differences, and weight of seed was found not to be correlated with potential level. Germination was recorded and found to be equal for all potential levels. However, all growth and yield records from the summer of 1944 must be discounted. There was a departure from normal precipitation of -9.65 during the months of May, June, July and August making the season so dry that plants made very poor growth and produced only about one fifth their normal yield. Nevertheless, Test No. 3 with the F_2 of crosses involving br, f_1 and bm_2 contributed useful information. Since the stock was segregating for these three recessive characters (one of which, br, is definitely deleterious since it dwarfs the plants by shortening all the internodes) it was suggested as a possibility that perhaps all the brachytic, or brachytic and brown mid-rib, might be concentrated in one potential level. This was not substantiated in field tests for the various genes and combinations of genes seemed to be distributed at random in the various levels (table 6).

TABLE 6

PERCENTAGE OF PLANTS SHOWING RECESSIVE CHARACTERS IN EACH POTENTIAL LEVEL

GENE	HIGH	LOW
bm_2	30	32
br	23	16
f_1	8	5
$bm_2 + br$	10	13
$bm_2 + f_1$	8	5
$br + f_1$	8	5
$bm_2 + br + f_1$	5	5

It was possible to obtain selfed seed on various potential levels in the No. 2 test($(Wf9 \times Hy)F_2$). Two lots of seed were selected at random from the selfed ears of the low potential plants and two from the seed of the high

potential plants. Potential levels had been measured in the F_2 seed in 1944, and the F_3 seed was considered to belong to the same potential levels without further testing. It was immediately apparent that there were considerable differences between the rows coming from the selfed seed of high potential plants and those coming from seed of low potential plants. The high potential plants were taller at every point during the growing season and in general were sturdier and leafier. Figure 5 gives a growth curve of height in inches plotted against time in days for the two high rows combined and the two low rows combined, and it seems certain that the progeny of high potential plants started faster and grew larger. Although results were combined in figure 5 it is well to mention that two high rows were very much alike in growth and superior to either low row.

We now have the intriguing situation in which e.m.f. differences found in the F_2 seeds are very markedly correlated with the growth characteristics of the F_3 plants. At the same time what has happened to the actual e.m.f. in the stocks between the F_2 and the F_4 which is the seed we now have on hand. The high division for the F_2 seed comprised those seeds with potentials of 17–30 millivolts while the low division had seeds with 0–2 millivolts as tested in 1944. After the large growth differences were obtained in 1945, the available F_3 and F_4 seed was tested. One selfed ear (F_3) from a high potential row had a mean of 34.20 millivolts while F_4 seed from this same ear measured 35.0. The F_2 seed from a low potential tested 26.1 millivolts, and no F_4 was available for measurement. This represents a considerable rise in F_3 and F_4 potentials over the F_2 for which it is difficult to account. However, the F_2 stock would be highly heterogeneous and capable of very considerable segregation of superior growth factors.

In connection with the differences between prime and equilibrium potential as reflected in percentage of germination, seed of the same sample of U. S. 13 as used in the prime potential tests (1945) was divided on the basis of its equilibrium potentials into high, medium and low groups. Two groups of untested seed were added to give five entries in a randomized block field test. There were 3 replications in this test which was intended solely to check germination. The percentage of germination for each entry was: untested (1) 93%, high 93%, untested (2) 93%, low 91%, medium 93%. The higher over-all percentage of germination was due to this test being planted in the middle of the summer with much less severe conditions than prevailed for the other test. But it serves to bear out other observations that the percentage of germination and the equilibrium potential are not correlated.

Summary.—For any maize there are two possible potential determinations. The first of these, the prime potential, is apparently highly correlated with seed viability, but with no other measured attribute of plant growth. The second of these potentials, the equilibrium potential, is not

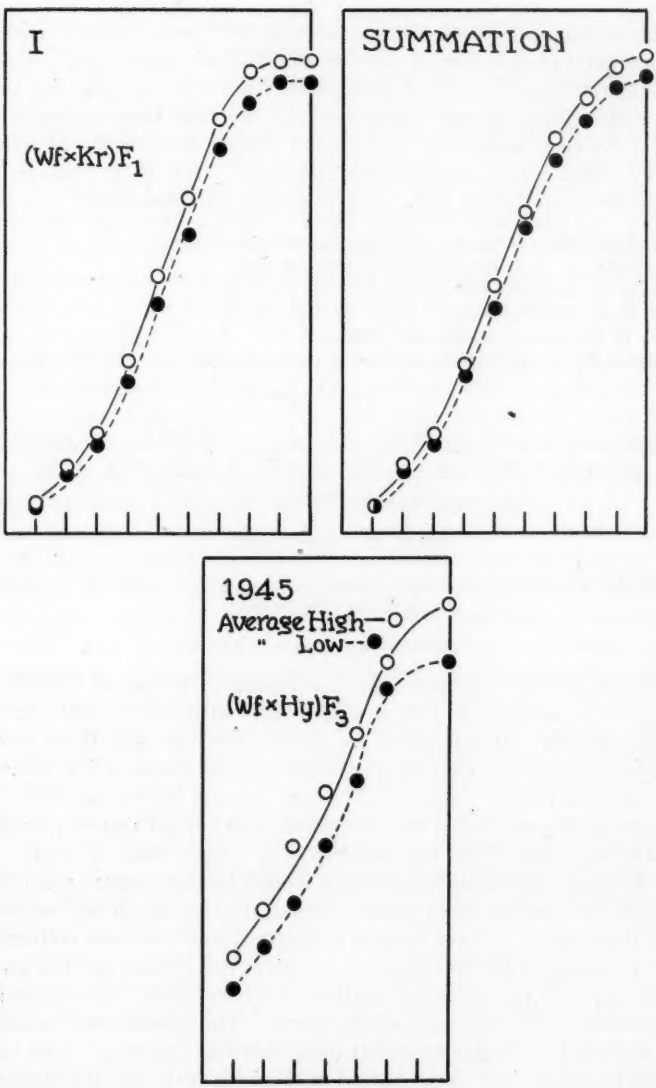

FIGURE 5

Graph of one additional line, plus a summation curve for all lines. Also growth curves from plants from F_3 plants where potential measurements were made on the F_2 seed.

correlated with seed viability but rather with the inherent genetic constitution of a plant since by use of the potentiometer and equilibrium potential determinations, one is enabled to segregate from a given population those seeds with superior growth characteristics. Further, these potential differences between seeds have been highly correlated with the growth of progeny which were one generation removed. For these reasons, the potentiometer may prove to be a useful tool for plant breeders.

* Graduate Fellow of the Eastern States Farmers' Exchange.
† The aid of Dr. C. I. Bliss is gratefully acknowledged.
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INTERSEXES DEPENDENT ON A MATERNAL EFFECT IN HYBRIDS BETWEEN DROSOPHILA REPLETA AND D. NEOREPLETA

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Drosophila repleta Wollaston and *D. neorepleta* Patterson and Wheeler are closely similar species, the former widely distributed and the latter known from Guatemala. It was found by Dr. E. Novitski that these species occasionally cross, and that the F_1 females sometimes give a few offspring when mated to repleta males (see Wharton 1942 and Sturtevant 1946).

I have found a sex-linked recessive white-eyed mutant type in *D. repleta* (actually not quite white, but retaining only a slight tinge of color). At least 5000 neorepleta females have been crossed to white repleta males (in a few cases the repleta males carried singed, another sex-linked recessive, rather than white). These matings included at least 500 mass cultures, of which 74 produced hybrid offspring—a total of 532 females and 635 males, all wild type for the sex-linked mutant characters used. The males had very narrow testes, and were wholly sterile. The females were variable; most of them had bristles somewhat reduced in size ("minute"), and many of them had three anal plates instead of the usual two—this last character suggesting intersexuality.

About 400 of these F_1 females were mated to white repleta males, and offspring were obtained from 34 of them—the total output being 70 wild-type females, 9 white females, 42 wild-type males and 58 white males. The expectation is for these classes to be equal; there is evidently a great

deficiency of white females. Of the 9 recorded in this class, notes made on two make it possible to be certain (in the light of later results, to be described below) that they were intersexes, and it is possible that some of the other 7 were also of this nature, as were probably a few of the white males.

The wild-type males from this first backcross resembled the F_1 males in having long narrow testes and in being sterile. The wild-type females, however, included some moderately fertile individuals, which were again crossed to white repleta males. Such successive backcrosses of wild-type females to white repleta males have now been continued through many generations, and have resulted in strains presumably pure repleta in composition except for a section of the *X*-chromosome near the locus of white. In some of these strains the double recessive, white singed, has been used instead of white; and in these strains all the wild-type males are sterile and have narrow testes, while all the white-singed males have normal testes and are as often fertile as are white-singed males with no neorepleta chromosomes in their immediate ancestry. Tests of the few crossovers between white and singed indicate that the locus of the narrow-testis gene (or genes) lies between white and singed, and very near white.

The wild-type females from the backcrosses, when mated to white repleta males, appeared to fall into two classes; some of them gave the four types of offspring in approximately equal numbers, whereas others continued to give a marked deficiency of white daughters. The daughters of the first type of female always repeated the first type of result; but the females of the second type commonly yielded few offspring, and their very existence as a distinct type was at first uncertain because of the confusing effects of sampling errors in the small families obtained from them. Finally, however, after four successive backcrosses, a single female of this type was obtained which was more fertile, and her descendants have retained this fertility. Presumably the neorepleta gene responsible for the unusual ratios was at first linked to another gene that decreases fertility in females largely repleta in constitution, and in this case the infertility gene was lost by crossing over. The later studies have all been carried out with descendants of this more fertile female. There are, however, enough fragmentary data to make it clear that the behavior is essentially the same in lines derived from other F_1 hybrids—i.e., the results are really due to genes derived from neorepleta, rather than to a mutation that occurred in this line.

From females of this more fertile line, backcrossed to white repleta males, a total of 33 wild-type daughters (of females giving a deficiency of white daughters) have been tested by white males. Of these, 16 gave approximately 1 + ♀ :1 w ♀ :1 + ♂ :1 w ♂; the remaining 17 all gave a deficiency of white females. The total counts from these 17 were: 472 + ♀, 5 w ♀, 63 w intersexes, 482 + ♂, 339 w ♂. The 5 white females pre-

sumably represent crossing over between the loci of white and of the critical gene in the X derived from neorepleta. Their nature can be more accurately determined when other sex-linked genes are introduced in the crosses; it is, however, already clear that the gene concerned is not the same as that responsible for the narrow testes of the hybrid males.

The intersexes are of an extreme type, with gonads very small (rudimentary ovaries in those cases where they were found at all); external genitalia missing or of abnormal male type; wings usually not expanded, and, when they are, usually with thickened veins; one or more (sometimes all four) scutellar bristles often absent. They are weak individuals—and evidently usually die before emergence. It is to be supposed that such preimaginal mortality is responsible for the difference in number of wild-type females (472) and white intersexes (63). While no systematic study has been made, white-eyed flies have been found dead in their puparia in such cultures.

These results suggest that there is an autosomal dominant gene, derived from neorepleta, that so conditions the eggs (before meiosis) that two repleta X-chromosomes result in the development of intersexes rather than females. Evidently the action comes before meiosis, and the autosomal gene in question may be absent in the intersexes themselves. This interpretation has been confirmed by tests of the white brothers of intersexes. Such males, when crossed to pure repleta females, gave normal offspring of both sexes; but some of their daughters (presumably half of the daughters from half of these males, though the data are not extensive enough to establish this), when mated to repleta males, gave only intersexes and males.

It will be seen that this last experiment shows that the intersexes are not dependent on the presence of neorepleta cytoplasm, since their mothers were offspring of pure repleta females. The experiment also shows that the autosomal gene from neorepleta has no phenotypic effect on females that have received it from their father. The result likewise agrees with the earlier ones in indicating that an individual with two repleta X's is intersexual whether or not it carries this gene, provided the gene was present in its mother.

Conclusions.—*D. neorepleta* carries an autosomal gene which, when present in single dose in a hybrid female, makes her eggs male in potentiality. This predisposition to maleness is only partially overcome by two repleta X's, and male-like intersexes result. One repleta X and one neorepleta X are sufficient to produce normal females. This autosomal gene is present in two doses in pure neorepleta eggs. Two neorepleta X's are sufficient to cause such eggs to develop into normal females; but in the F_1 hybrids, having one neorepleta X and one from repleta, some of these eggs develop into females with male-like anal plates.

Presumably the essential sex-determining mechanism is the same in the two species, but both the male-determining action of the autosomal gene and the female-determining action of the X are stronger in neorepleta.

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THE MECHANISM OF POSITION EFFECT—EXPERIMENTS ON THE PHENOTYPIC EXPRESSION OF POSITION EFFECTS IN RELATION TO CHANGES IN PAIRING OF NEIGHBORING CHROMOSOME REGIONS

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In a previous paper[1] we discussed two alternative types of interpretation of the phenomenon of position effect, and expressed a predilection for one of these alternatives, partly on the grounds that it seemed to lend itself more readily than the other to experimental tests. The hypothesis as to the mechanism of position effect that we thereupon elaborated was closely related to some early suggestions of Muller.[2, 3] Briefly stated, it postulated the following chain of events. In an organism such as Drosophila, where somatic pairing occurs, chromosomal aberrations change the pairing relationships of the chromosome regions adjacent to the breaks, or facing the breaks. The forces which bring about pairing may thus achieve a new distribution on either side of a gene located near to or facing a break. Such a change in pairing forces might subject this gene to a changed condition of stress. Now, if we visualize the gene as a complex folded protein molecule, or part of such a molecule, the specific activity of which is determined by the spatial configuration of specific groups on its surface, then a change of stress might be expected to lead to a change in the degree of extension of the folded protein, hence to a change in the spatial relationship of the active group, and so finally to a change in the specific activity of the gene, which change may be manifested phenotypically as a position effect.

We concluded that if this hypothesis were valid it should be possible, given an already existing position effect, to modify its phenotypic expression by further changing the pairing relationships of the chromosome regions in the immediate neighborhood of the affected gene. Moreover, it seemed clear that it would not be necessary for this purpose to alter the

already aberrant chromosome in which the affected gene itself was located, but that the change in pairing, and therefore the phenotypic change, should be obtained equally well by introducing an aberration in the homologous chromosome, opposite and close to the affected gene.

The experiments presented here were designed to serve this purpose.

Experiments and Results.—For the position effects, the white (*w*) locus was chosen, and three white-mottled stocks (w^{m4}, w^{m5} and $w^{258\text{-}18}$) were used.[4] The first of these (w^{m4}) involves an inversion in the *X*-chromosome, with one break to the left of w^+ and the other in the heterochromatic region of the chromosome, while each of the other two stocks carries a translocation between the *X*-chromosome and chromosome 4, the break in *X* being to the right of w^+. In all three stocks the males and homozygous females are viable, and the white-mottled eyes are characteristic of these flies as well as of females heterozygous for the rearrangement and for a normal *X* carrying the mutant allele *w*.

In order to combine the position effects with an aberration adjacent to the *w* locus in the opposite chromosome, deficiencies were used, and a comparison was made of heterozygous females ($w^m/y\ Hw\ w$) and females ($w^m/Df\ w$) in which an *X*-chromosome with a deficiency next to the *w* locus was substituted for the *y Hw w* chromosome, the dominant *Hw* being used as a marker to distinguish the two types (see figure 1; here, and throughout this paper, the following symbols are used: *w* = white, w^m = white-mottled, *y* = yellow, *Hw* = hairy wing, *Df* = deficiency).

Three deficiency stocks were used, $w^{258\text{-}45}$, $w^{258\text{-}48}$ and $w^{258\text{-}14}$, in which there are deficiencies of one band, 5 bands and 13 bands, respectively, immediately to the left of, but not including, the *w* locus (assumed to be associated with band 3 C 2.3)[4] which was represented by the recessive *w*.

Crosses of the general type $Df\ w/y\ Hw\ w$ ♀ × w^m ♂ were made, so that the $w^m/Df\ w$ and $w^m/y\ Hw\ w$ F_1 females to be compared developed in the same bottle under the same conditions. Flies were raised at 25°C. except in a few of the initial experiments, which were run at variable room temperatures, probably averaging 26° or 27°C.

The $w^m/Df\ w$ flies and the $w^m/y\ Hw\ w$ controls were classified individually as to eye-color in four arbitrary groups—light, light intermediate, dark intermediate and dark. As there was no objective control (such as a color chart) for this system of classification, the standards adopted for the four groups may have varied somewhat from time to time. Nevertheless, the method was satisfactory for comparing flies at any one time, and the conclusions drawn from comparison of $w^m/Df\ w$ flies and $w^m/y\ Hw\ w$ controls from the same bottle are valid.

In eight out of the nine possible combinations between the three white-mottled and the three deficiency stocks, a significant difference[5] was observed in the distribution of the $w^m/Df\ w$ and $w^m/y\ Hw\ w$ flies in the four

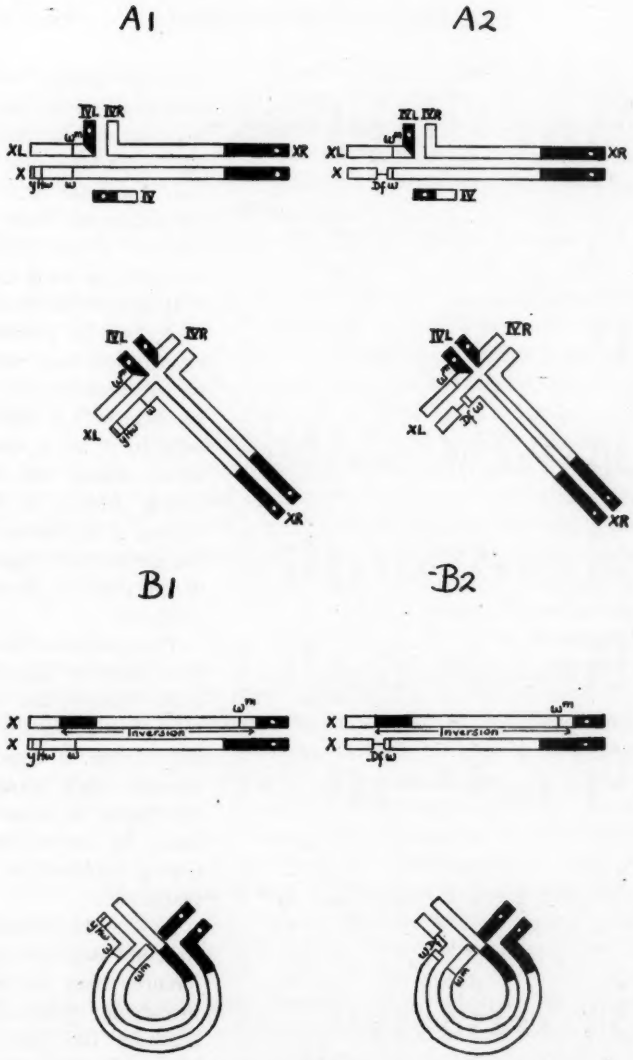

FIGURE 1

Diagram showing X-chromosomes (first and third lines) and their pairing (second and fourth lines). A represents combinations with the w-mottled stocks $w^{258\text{-}18}$ and w^{m5}; B, combinations with the w^{m4} stock. $A1$ and $B1$, controls; $A2$ and $B2$, substitution of an X-chromosome with a deficiency (single line) for a normal X. Maximum pairing is assumed for euchromatic regions (double line), but the non-homologous pairing of heterochromatic regions (solid black) is not indicated.

TABLE 1

EYE-COLORS OF $w^m/Df\ w$ FEMALES AS COMPARED WITH THE CORRESPONDING $w^m/y\ Hw\ w$ CONTROLS

w^m combination	$\frac{w^{258\text{-}18}}{w^{258\text{-}45}}$	$\frac{w^{m5}}{w^{258\text{-}45}}$	$\frac{w^{m4}}{w^{258\text{-}45}}$	$\frac{w^{258\text{-}18}}{w^{258\text{-}45}}$	$\frac{w^{m5}}{w^{258\text{-}45}}$	$\frac{w^{m4}}{w^{258\text{-}45}}$	$\frac{w^{258\text{-}18}}{w^{258\text{-}14}}$	$\frac{w^{m5}}{w^{258\text{-}14}}$	$\frac{w^{m4}}{w^{258\text{-}14}}$
Df w	0	0	0	0	0	4	0	0	22
Light intermediate	16	59	22	2	0	53	0	0	96
Dark intermediate	175	249	275	17	3	43	1	2	93
Dark	36	102	192	27	84	10	24	65	73
w^m control	$\frac{w^{258\text{-}18}}{y\ Hw\ w}$	$\frac{w^{m5}}{y\ Hw\ w}$	$\frac{w^{m4}}{y\ Hw\ w}$	$\frac{w^{258\text{-}18}}{y\ Hw\ w}$	$\frac{w^{m5}}{y\ Hw\ w}$	$\frac{w^{m4}}{y\ Hw\ w}$	$\frac{w^{258\text{-}18}}{y\ Hw\ w}$	$\frac{w^{m5}}{y\ Hw\ w}$	$\frac{w^{m4}}{y\ Hw\ w}$
Light	0	0	0	3	5	1	3	5	20
Light intermediate	10	20	3	13	35	18	37	8	99
Dark intermediate	230	200	215	58	98	64	119	21	122
Dark	34	213	282	52	160	60	89	228	162
Temperature	25 °C.	25 °C.	25 °C.	Room temp.	Room temp.	Room temp.	25 °C.	25 °C.	Room temp.
χ^2	4.5	63.1	38.6	4.2	53.4	55.6	34.1	5.3	17.6
n	2	2	2	1	2	2	1	1	3
P	0.20–0.10	<0.01	<0.01	<0.05	<0.01	<0.01	<0.01	<0.05	<0.01

eye-color classes. In some combinations the flies with deficiencies were generally lighter in eye-color than the controls, in other combinations they were darker. These differences are summarized in table 1.

It is obvious from this table that in general our expectation was fulfilled: on substitution of a chromosome with a deficiency near to *w* for a chromosome which was structurally normal in the *w* region, a modification of the phenotypic expression of the position effect was obtained.

There remains the question whether the phenotypic changes are indeed related to changes in pairing in the *w* region, or whether they should be attributed to some other cause, for any of the following explanations seem plausible.

1. As the stocks used were not isogenic, the differences may be due to modifying genes distinguishing the three deficiency chromosomes from the *y Hw w* chromosome. These modifiers could be located (*a*) in the *X*-chromosome, but not closely linked with the *w* locus; (*b*) in the deficient region of the *X*-chromosome (and

thus closely linked with *w*) and affecting the total balance of + and − modifying genes through their presence or absence; or (*c*) closely linked with *w*, but not in the region of the deficiency.

2. The differences could be due to differences in competitive action of the *w* alleles in the deficient chromosomes, like that postulated by Stern[6] in the case of *cubitus interruptus* alleles.

These possibilities will be considered in order.

1. (*a*) To make all stocks as isogenic as possible, they were all outcrossed repeatedly to a single stock of *y Hw w*. The crosses were also made in such a way as to select for viability of the $w^m/Df\ w$ combinations, which in some cases was rather poor. The procedure adopted was to obtain each $w^m/Df\ w$ combination, outcross it to *y Hw w* males, and in the next generation breed together $Df\ w/y\ Hw\ w$ females and w^m males. Then the $w^m/Df\ w$ females were selected and outcrossed to *y Hw w* males a second time, and so on. This procedure should have two results: (1) replacement of autosomes of the w^m and *Df w* stocks by those of the *y Hw w* stock, and (2) replacement by crossing-over of parts of the *Df w* and eventually of the w^m *X*-chromosomes by the *y Hw w X*. Although the *y Hw w* stock itself was not strictly isogenic, it should thus serve to give both $w^m/Df\ w$ and $w^m/y\ Hw\ w$ flies a similar range of variation in genetic background, with the added reservation that in regions close to the breaks the substitution of genes from the *y Hw w* stock could not be expected at all, owing to the low frequency of crossovers in these regions.

The extent to which different combinations were outcrossed and the results of the outcrossing, are shown in table 2. The most important of these results is that of the $w^{m5}/w^{258\text{-}14}$ combination. A comparison of tables 1 and 2 shows that though at first the eyes of $w^{m5}/w^{258\text{-}14}$ were darker than those of $w^{m4}/y\ Hw\ w$, after eight generations of outcrossing $w^{m5}/w^{258\text{-}14}$ became lighter than $w^{m5}/y\ Hw\ w$. This must mean that the original $w^{258\text{-}14}$ *X*-chromosome carried modifiers for dark eye-color which were effectively removed by crossing-over in the course of several generations of outcrossing to *y Hw w*, and that the difference in the opposite direction which then became apparent had been masked by these modifiers at an early stage, before their substitution by genes from the *y Hw w X*-chromosome. This later difference therefore cannot be attributed to modifiers in any other part of the *X*-chromosome than the close neighborhood of *w*.

Other combinations which were outbred for several generations continued to show the same difference between the $w^m/Df\ w$ and $w^m/y\ Hw\ w$ flies that appeared at the first cross.

In two cases ($w^{258\text{-}18}/w^{258\ 48}$ and $w^{m4}/w^{258\text{-}48}$ combinations) after crossovers had occurred between the *Df w* and *y Hw w* chromosome to the left of the deficiency, the crossover chromsome consisting of the left end of the deficiency chromosome (but without the deficiency) and the right end

of the *y Hw w* chromosome (without *y Hw*) was tested by making up the combination *crossover/y Hw w* and mating to w^{m4} or w^{258-18} males. The distribution of eye-colors in w^{m}/*crossover* and w^{m}/*y Hw w* females of the next generation did not differ significantly, showing that the left end of the deficient w^{258-48} *X*-chromosome was not distinguishable from that of the *y Hw w* chromosome with regard to modifiers of the position effects.

It may be concluded from the outbreeding experiments that the final differences observed between w^{m}/*Df w* and w^{m}/*y Hw w* flies were not due to any difference in such modifiers as could be separated from the *w* region by crossing over. This conclusion is substantiated, as far as the region to the left of *w* is concerned, by the tests of crossovers just described.

TABLE 2

EYE-COLORS OF w^{m}/*Df w* FEMALES AS COMPARED WITH THE CORRESPONDING w^{m}/*y Hw w* CONTROLS, AFTER OUTCROSSING TO *y Hw w* 25°C.

$\frac{w^{m}}{Df\ w}$ combination	$\frac{w^{m5}}{w^{258-45}}$	$\frac{w^{m4}}{w^{258-45}}$	$\frac{w^{258-18}}{w^{258-45}}$	$\frac{w^{m4}}{w^{258-45}}$	$\frac{w^{258-18}}{w^{258-14}}$	$\frac{w^{m5}}{w^{258-14}}$	$\frac{w^{m4}}{w^{258-14}}$
Light	0	0	0	0	0	0	0
Light intermediate	0	1	0	14	0	10	25
Dark intermediate	150	80	2	130	1	99	151
Dark	386	250	64	165	21	222	59
$\frac{w^{m}}{y\ Hw\ w}$ control	$\frac{w^{m5}}{y\ Hw\ w}$	$\frac{w^{m4}}{y\ Hw\ w}$	$\frac{w^{258-18}}{y\ Hw\ w}$	$\frac{w^{m4}}{y\ Hw\ w}$	$\frac{w^{258\ 18}}{y\ Hw\ w}$	$\frac{w^{m5}}{y\ Hw\ w}$	$\frac{w^{m4}}{y\ Hw\ w}$
Light	0	0	0	0	1	0	2
Light intermediate	0	0	9	2	29	0	12
Dark intermediate	74	48	98	59	86	38	121
Dark	633	323	156	325	76	355	105
No. of generations outcrossed	10	10	13	14	2	8	7
χ^2	63.3	15.5	31.1	80.5	24.7	60.1	20.2
n	1	1	1	2	1	1	2
P	<0.01	<0.01	<0.01	<0.01	<0.01	<0.01	<0.01

(*b*) The loss of dominant modifiers in the deficient chromosomes could result in a difference in phenotype of flies carrying these chromosomes as compared with those having a complete *y Hw w* chromosome. Inspection of the tables, however, shows that the same deficiency (e.g., w^{258-48}) may modify different position effects in opposite directions. In other words, on the assumption of a change in balance of dominant modifiers, we would also have to assume that these modifiers have a different and specific action on different position effects. While this interpretation does not appear probable to us, it obviously cannot be excluded altogether.

There is another possibility, however, namely that the deficiencies uncover recessive modifiers in the opposite chromosome, near to the affected *w* locus. In this case, the reaction of different position effects to the same deficiency would be attributable not to specific action of the same

modifiers, but to the presence of different modifiers closely linked with the affected *w* loci of the different white-mottled stocks.

If this were so, we would assume for instance that in the w^{258-18}/w^{258-48} combination, the deficiency uncovers recessive modifiers which darken the eye-color, while in w^{m4}/w^{258-48} one or more recessive modifiers for lighter color are effective. These modifiers, then, should also be effective in the hemizygous males and homozygous females of the position effect stocks; but this is not the case, for w^{258-18} and w^{m4} ♂s were scored after 13 and 12 generations of outcrossing, respectively, and in both cases (not in w^{m4} only) proved to be lighter than the corresponding w^{258-18}/y *Hw w* or w^{m4}/y *Hw w* ♀s.

We consider, therefore, that this possibility can be discarded.

It may be added that the probability that differences are due to modifiers in the deficient regions is considerably reduced by the fact that such differences are obtained even with the single band deficiency of w^{258-45}.

(*c*) It will be seen that the assumption of very closely linked modifiers outside of the deficiency differing from those of the *y Hw w* chromosome again requires the additional assumption of specific and opposite action of these genes on different position effects. We are left in doubt as to this possibility, as in the case of possibility 1(*b*) above.

2. Stern[6] has obtained results with alleles at the *cubitus interruptus* (*ci*) locus which he interprets on the supposition that some of these alleles compete with one another for the use of a common substrate, and that differences in ability to combine with this substrate or to convert it into a new end-product are the cause of different degrees of phenotypic expression of the *cubitus interruptus* effect in flies carrying different combinations of alleles.

In order to test the *w* alleles in our deficiency chromosomes for such a competitive effect, females of the constitution w^{258-14}/y *Hw w*,w^{258-45}/y *Hw w* and w^{258-48}/y *Hw w* were mated with apricot (w^a) males, and in each case a comparison was made between the w^a/Df *w* and w^a/y *Hw w* progeny.

It was impossible to distinguish between the two types in any case, so that there is no evidence that any of the deficiencies used have any effect on the expression of the w^a allele.

Summary and Conclusion.—On the assumption that pairing conditions affect the manifestation of position effect, experiments were performed in which the eye-colors of flies, carrying a white-mottled *X*-chromosome and either a normal unbroken *X* or one with a deficiency next to the *w* locus, were compared. The eye-colors appear to be different. While these results can be regarded as supporting the hypothesis which the experiments were designed to test, it cannot be completely excluded at present that they are due to the action of modifiers, or to competition between alleles.[7]

[1] Ephrussi, B., and Sutton, E., these PROCEEDINGS, **30**, 183–197 (1944).

[2] Muller, H. J., *Fifteenth Int. Physiol. Congr., Leningrad* (1935).

[3] Muller, H. J., *Cold Spring Harbor Symp. Quant. Biol.*, **9**, 151–165 (1941).

[4] Bridges, C. B., and Brehme, K. S., "The Mutants of *Drosophila melanogaster*," *Carnegie Inst. Washington, Pub.* **552**, Washington, D. C. (1944).

[5] The χ^2 test was applied, classes being combined whenever the expected numbers in a single class were less than 5.

[6] Stern, C., *Genetics*, **28**, 441–475 (1943).

[7] This paper was read in manuscript by Drs. H. B. Glass, R. F. Kimball, B. McClintock, H. J. Muller and B. H. Willier, to whom we wish to express our appreciation.

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*THE DISTORTION OF ANGLES IN GENERAL CARTOGRAPHY**

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1. Azimuthal Curves.—Let a surface Σ be mapped in a point-to-point fashion upon a plane π with cartesian coördinates (x, y) such that point s of Σ and π will correspond if they are represented by the same (curvilinear) coördinates. Recently[1] we have introduced the term *cartogram*, to denote any particular mapping of a surface Σ upon a plane π. The cartogram is conformal or general according as the transformation is or is not conformal.

We define the azimuthal ratio α in the following manner. Let $d\theta$ denote the angle between two infinitesimally consecutive directions at a fixed point $p(x, y)$ on the plane π, and $d\Theta$ the angle between the corresponding consecutive directions at the associated fixed point $P(x, y)$ on the surface Σ. By the *azimuthal ratio* α, we shall mean the value of the fraction: $\alpha = d\Theta/d\theta$. That is, it is the instantaneous rate of change of the inclination at the point $P(x, y)$ on the surface Σ with respect to the inclination at the corresponding point $p(x, y)$ on the plane π.

Our azimuthal ratio α is a function of the lineal element (x, y, y'). It is independent of the slope $y' = dy/dx$ if, and only if, the cartogram is conformal, in which case $\alpha = 1$.

An *azimuthal* curve is the locus of a point on the plane π (or on the surface Σ) along which the azimuthal ratio α does not vary. *In a general cartogram, there are* ∞^2 *azimuthal curves.* (In this connection, conformal cartograms[2] are of no interest since every curve is azimuthal because of the fact that $\alpha = 1$.)

In the present article, we shall derive some of the geometrical properties of systems of ∞^2 azimuthal curves. Also we shall compare and contrast these with our properties of scale curves which were developed elsewhere.[3]

2. The formula for the Azimuthal Ratio α.—To derive this formula, we

proceed as follows: The square of the linear element dS on the surface Σ, is

$$dS^2 = E(x, y)dx^2 + 2F(x, y)dxdy + G(x, y)dy^2, \qquad (1)$$

where $H^2 = EG - F^2 > 0$. On the other hand, the square of the linear element ds on π is: $ds^2 = dx^2 + dy^2$.

The angle $\Delta\Theta$ between any two directions y'_1 and y'_2 at the fixed point $P(x, y)$ on the surface Σ is given by the equation

$$\tan \Delta\Theta = \frac{H(y'_2 - y'_1)}{E + F(y'_1 + y'_2) + Gy'_1y'_2}. \qquad (2)$$

The angle $\Delta\theta$ between the two corresponding directions y'_1 and y'_2 at the associated fixed point $p(x, y)$ on the plane π, is given by the formula: $\tan \Delta\theta = (y'_2 - y'_1)/(1 + y'_1y'_2)$.

Upon dividing the two preceding equations defining the increments of angles and then letting the slope y'_2 approach the slope y'_1, we obtain *the following formula for the azimuthal ratio*

$$\alpha = \frac{d\Theta}{d\theta} = \frac{H(1 + y'^2)}{E + 2Fy' + Gy'^2}. \qquad (3)$$

This formula gives the rate of distortion of the angle in any general cartogram.

Noting that the scale σ is defined by $\sigma = ds/dS$, it follows that *the azimuthal ratio α is the product of H by the square of the scale σ. That is,*

$$\alpha = H\sigma^2. \qquad (4)$$

3. The Differential Equation of Second Order Defining the System of ∞^2 Azimuthal Curves.—Along any curve of this system, α = const. Thus upon eliminating the constant α from the equation (3) by differentiation, we find that *the differential equation of our ∞^2 azimuthal curves is*

$$y'' = \frac{(1 + y'^2)[E_x + (E_y + 2F_x)y' + (2F_y + G_x)y'^2 + G_yy'^3 - \frac{1}{H}(H_x + y'H_y)(E + 2Fy' + Gy'^2)]}{2[-F + (E - G)y' + Fy'^2]}. \qquad (A)$$

It follows that not every system of ∞^2 curves can represent the azimuthal curves of a cartogram since y'' is of special algebraic character in the first derivative y'.

4. Comparison of the Systems of Azimuthal and Scale Curves.—A scale curve is the locus of a point along which the scale $\sigma = ds/dS$, does not vary. The differential equation of the ∞^2 scale curves is

$$y'' = \frac{(1+y'^2)[E_x + (E_y + 2F_x)y' + (2F_y + G_x)y'^2 + G_y y'^3]}{2[-F + (E-G)y' + Fy'^2]}. \quad (S)$$

Upon contrasting the system (A) of azimuthal curves and the system (S) of scale curves, we observe that they are of the same general algebraic structure in y'.

The cuspidal directions (corresponding to zero radius of curvature) for both systems (A) and (S) are along the Tissot characteristic net

$$Fy'^2 + (E-G)y' - F = 0. \quad (5)$$

This is the unique orthogonal net on the plane π which by the general cartogram is converted into an orthogonal net on the surface Σ.

The system (A) of azimuthal curves and the system (S) of scale curves are identical if, and only if, the cartogram is an equiareal map followed by a magnification.

For the general cartograms not of the above type, *there are 5 ∞^1 curves which are simultaneously azimuthal and scale curves.* These are (a) the $2\infty^1$ minimal lines on the plane π, (b) the $2\infty^1$ minimal curves on the surface Σ, and (c) the ∞^1 curves $H(x, y) = \text{const}$.

Through a fixed lineal element of the plane, there pass, in general, a single azimuthal curve and a single scale curve. Let K_A be the curvature of the azimuthal curve, and let K_S be that of the scale curve, both of which are calculated at this given lineal element.

By (A) and (S), it is found that the ratio K_A/K_S, is determined by the formula

$$1 - \frac{K_A}{K_S} = \frac{(H_x + H_y y')(E + 2Fy' + Gy'^2)}{H[E_x + (E_y + 2F_x)y' + (2F_y + G_x)y'^2 + G_y y'^3]}. \quad (6)$$

Thus the ratio K_A/K_S is a rational function of third degree in y' with coefficients functions of (x, y).

In special cases, this ratio K_A/K_S may be a rational function of second or first degree in y' with coefficients functions of (x, y). It may be even independent of the slope y', and thus depend only on the position of the point.

5. *Cartograms for Which the Azimuthal Curves Are Straight.*—In our previous work, we found a new class of surfaces for which there exist cartograms whose ∞^2 scale curves are all straight lines. Concerning the analogous problem for azimuthal straight lines, it is found that any arbitrary surface Σ may be mapped upon a plane π such that the azimuthal curves are all straight lines. We prove the following result.

If a cartogram of an arbitrary surface Σ is such that the azimuthal curves coincide with the totality of ∞^2 straight lines, then it is a conformal representation of Σ upon the plane π, followed by an affine (not a motion) transformation in π.

For if the system (A) of ∞^2 azimuthal curves consists of straight lines only, it may be shown that

$$\frac{E}{H} = a_0y^2 + 2a_1y + a_2, \frac{F}{H} = -a_0xy - a_1x + b_1y + c_2, \frac{G}{H} = a_0x^2 - 2b_1x + b_2. \tag{7}$$

Substitute these into the condition: $H^2 = EG - F^2 > 0$. We obtain the relations

$$a_2b_2 - c_2^2 = 1, a_0a_2 - a_1^2 = a_0c_2 - a_1b_1 = a_0b_2 - b_1^2 = a_1c_2 - a_2b_1 = a_1b_2 - b_1c_2 = 0. \tag{8}$$

From these, we deduce that $a_0 = a_1 = b_1 = 0$.

Upon dropping the subscripts, it follows from (7) and the preceding conditions that the linear element (1) must be written in the form

$$dS^2 = H(adx^2 + 2cdxdy + bdy^2), \tag{9}$$

where (a, b, c) are constants such that $ab - c^2 = 1$, and H is a positive function of (x, y).

The preceding remarks complete the proof of the above italicized statement.

If in a cartogram, the azimuthal curves and the scale curves all coincide with the totality of ∞^2 straight lines, then the surface Σ is developable and the transformation is an unrolling of Σ upon the plane π followed by an affinity in π.

This is obtained as a corollary of the work outlined above. For upon imposing the condition that the scale curves of the cartogram defined by (9) be all straight lines, we find that the function H is a non-zero positive constant.

The result concerning straight azimuths is strikingly different from the corresponding theorem about straight scales. Any surface Σ can be mapped upon the plane π with straight azimuths but only a very special class of surfaces Σ can be represented upon the plane π with straight scales.

* Presented to the American Mathematical Society, April, 1946. The following papers are all by Kasner and De Cicco.

[1] "Scale Curves in Cartography," *Science*, **98**, 324–325 (1943), and *Science News Letter*, March 25, 1944.

[2] "Scale Curves in Conformal Maps," these PROCEEDINGS, **30**, 162–164 (1944). "Geometry of Scale Curves in Conformal Maps," *Am. Jour. Math.*, **67**, 157–166 (1945). "Conformal Maps with Isothermal Systems of Scale Curves, *Ibid.*, **68**, 137–166 (1946).

[3] "Scale Curves in General Cartography," these PROCEEDINGS, **30**, 211–215 (1944). See *Am. Jour. Math.*, **68**, 66–76 (1946). *Scale Curves and Cartograms*, Boletino del Instituto de Matematicas del Universidad de Litoral, Rosario, Argentina (1946). One of our results on the converse of Ptolemy's theorem has been given by Hilton, 1928.

A GENERALIZATION OF THE WIENER-HOPF INTEGRAL EQUATION*

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I. Introduction.—In this note we outline some results which we have obtained by generalizing the Wiener-Hopf integral equation.[1] This integral equation is of the form

$$f(x) = \lambda \int_0^\infty K(x - y)f(y)dy \qquad x > 0 \qquad (1.1)$$

where $K(x)$ is a known function. If the kernel $K(x)$ possesses a bilateral Fourier transform

$$\int_{-\infty}^{\infty} e^{-i\omega x}K(x)dx$$

which exists in some strip in the complex ω plane $\beta < \Im m\omega < \beta'$, one can obtain the so-called "fundamental solutions"[2] of equation (1.1) with the aid of certain regularity properties of the Fourier transform. We generalize (1.1) in a twofold fashion. First, a method is given which will enable one to write a formal solution to the integral equation

$$f(x) = \lambda \int_0^\infty N(x + y)f(y)dy \qquad x > 0 \qquad (1.2)$$

where now $N(x)$ is integrable in any interval which excludes the origin, $0(e^{-x})$ as $x \to \infty$ and $0(1/x)$ as $x \to 0$. The plus sign in the kernel now changes the entire mode of attack. While Fourier transform methods still play a dominant rôle in our work, we cannot solve the equation (1.2) in one fell swoop as we do equation (1.1). Instead we find it necessary to take advantage of the special assumptions which we impose upon $N(x)$ and decompose equation (1.2) into an infinite sequence of bilateral faltung equations, each one of which depends on the solution of the previous one. These we can solve step by step and our final answer appears as an infinite series of integral operators acting on a known function which has been determined in the course of our work. Thus a formal solution can be given for the above-described class of integral equations and this solution depends only upon the operation of integration. The decomposition is in some respects reminiscent of the method of successive approximations employed in the solution of non-homogeneous linear integral equations.

In the second place, under special conditions which we shall discuss elsewhere, the class of integral equations

$$\int_0^\infty N_1(x - y)f(y)dy + \int_0^\infty N_2(x + y)f(y)dy = \lambda f(x) \qquad x > 0 \qquad (1.3)$$

may be reduced to an integral equation of the form (1.2). Here $N_1(x)$ and $N_2(x)$ are given functions of x with special growth properties at infinity and are otherwise integrable for infinite x.

II. Formal Solution of Equation (1.2).—Specifically we are concerned with finding those solutions of the equation (1.2) which are of the order x^α, $-1 < \alpha < 0$ for $x \to 0$. λ is a parameter which takes on real values between 0 and $1/\pi$ if we are to obtain real solutions which have the above-described order at the origin. Since $N(x)$ is asymptotic to e^{-x} as $x \to \infty$, we obtain a solution of (1.2) which is of the same order as $x \to \infty$. We show that α may be determined from a characteristic equation, which is of course expressed in terms of λ.

Our technique is now the following. Since $f(x)$ approaches zero exponentially, as x becomes infinite and since $N(x)$ is singular as x approaches zero, we take out the dominant part of the solution by decomposing equation (1.2) as follows. We rewrite equation (1.2) as the following system of integral equations

$$f_0(x) = \lambda \int_0^\infty \frac{f_0(y)}{x+y} dy \tag{2.1}$$

$$f_n(x) = \lambda \int_0^\infty \frac{f_n(y)dy}{x+y} + \lambda \int_0^\infty \left[N(x+y) - \frac{1}{x+y} \right] f_{n-1}(y)dy; \quad n = 1, 2, \ldots \tag{2.2}$$

where

$$f(x) = \sum_{n=0}^{\infty} f_n(x).$$

Equation (2.1) has been studied extensively in the literature and is known possess the solutions x^α, $x^{-1-\alpha}$, where $\sin \alpha\pi = -\lambda\pi$. It has been further shown by Hardy and Titchmarsh, that the only two linearly distinct solutions of (2.1) are of this form.

If we put $x = e^t$, $y = e^s$ equations (2.2) assume the form of bilateral faltung equations, that is

$$f_n(e^t)e^{t/2} = \lambda \int_{-\infty}^{\infty} \frac{f_n(e^s)e^{s/2}ds}{e^{(t-s)/2} + e^{(t-s)/2}} + \lambda e^{t/2} q_n(e^t); \qquad n = 1, 2, \ldots \tag{2.3}$$

where

$$q_n(x) = \int_0^\infty \left[N(x+y) - \frac{1}{x+y} \right] f_{n-1}(y)dy.$$

Equations (2.3) can be solved with the bilateral Fourier transform theorem in the complex domain. The solution can be exhibited explicitly with the

aid of an integral operator which we do not write out here explicitly, but denote by P, that is,

$$f_n(x) = \lambda q_n(x) + Pq_n \tag{2.4}$$

where now it is understood that

$$Pq_n = \int_0^\infty P(x, t)q_n(t)dt$$

and $P(x, t)$ is a known function of x and t. If we denote the integral operations

$$\int_0^\infty N(x + y)f_n(y)dy \qquad \text{by } Nf_n$$

and

$$\int_0^\infty \frac{f_n(y)dy}{x + y} \qquad \text{by } Lf_n$$

we observe that the equations (2.2) can be rewritten as

$$f_n(x) = \lambda Lf_n + \lambda Nf_{n-1} - \lambda Lf_{n-1} \tag{2.5}$$

and $q_n(x)$ as

$$q_n(x) = Nf_{n-1} - Lf_{n-1}.$$

We then get immediately,

$$f_n(x) = \lambda(1 + P)(\lambda NP + \lambda N - P)^{n-1}q_1$$

where the exponent $n - 1$ indicates that the operation within the parentheses is to be applied $n - 1$ times and furthermore the operators N and P do not commute. The final formal solution is then

$$f(x) = f_0(x) + \sum_{n=1}^{\infty} \lambda(1 + P)(\lambda NP + \lambda N - P)^{n-1}q_1$$

and this indeed satisfies the integral equation (1.2).

III. An Example.—If we now apply the previously described technique to the integral equation

$$f(x) = \lambda \int_0^\infty \frac{e^{-(x+y)}f(y)}{x + y}dy$$

we find that $f(x)$ is $K_{\alpha+1/2}(x)/\sqrt{x}$, where now α is defined by the constraining equation $\lambda\pi = -\sin \pi\alpha$, $0 < \lambda\pi < 1$ and $-1 < \alpha < 0$. The function $K_{\alpha+1/2}(x)$ is a modified Bessel function of the second kind which is invariant under the substitution of replacing α by $-\alpha - 1$. It is clear that $f(x)$ is of the order x^α or $x^{-\alpha-1}$ for small x, depending upon which term dominates. For x large and positive, $f(x)$ is of the order e^{-x}.

A word of description as to how we succeeded in obtaining the solution in closed form is in order. The procedure indicated in Section II leads to an infinite expansion of iterated integral operators. For this particular example, this expansion can be shown to be the solution of a Wiener-Hopf integral equation, which of course can be solved explicitly. There is much hope that the Wiener-Hopf technique will play an important rôle in the solution of integral equations of the form (1.2) when the exponential character of the kernel $N(x + y)$ is exhibited explicitly as it is in the above example.

* Presented at the Chicago Meeting of the American Mathematical Society, November 24, 1945.

† Pro tem on leave of absence. At present at the Radiation Laboratory, The Massachusetts Institute of Technology.

[1] Paley and Wiener, "Fourier Transforms in the Complex Domain," *Colloq. Pub. Am. Math. Soc.*, Chapter 4, 1934.

[2] Paley and Wiener, loc. cit., Chapter 4.

ON CLASSES OF DIOPHANTINE EQUATIONS OF HIGHER DEGREES WHICH HAVE NO SOLUTIONS

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This paper is concerned with the isolation of certain classes of Diophantine equations which have no solutions or no non-zero solutions, using the theory of congruences. Let $f(x_1, x_2, \ldots, x_s)$ and $g(y_1, y_2, \ldots, y_t)$ be polynominals in $x_1, x_2, \ldots, x_s$ and $y_1, y_2, \ldots, y_t$ with integral coefficients. Then if m is an integer and

$$f(x_1, x_2, \ldots, x_s) \equiv 0 \pmod{m} \tag{1}$$

has no solutions in $x_1, x_2, \ldots, x_s$, then it is obvious that the equation

$$f(x_1, x_2, \ldots, x_s) + mg(y_1, y_2, \ldots, y_t) = 0 \tag{2}$$

has no solution in integers $x_1, x_2, \ldots, x_s; y_1, y_2, \ldots, y_t$. However, if $f(x_1, x_2, \ldots, x_s)$ is homogeneous, then the congruence (1) has the solution $x_1 = x_2 = \ldots x_s = 0$, so that this principle cannot be employed immediately for homogeneous equations. Also it may be shown by the use of congruences that the equation

$$v^3 - u^4 = 10$$

has no solutions, but its relation to an equation of the form (2) is not ob-

vious. Here we shall establish types of congruences which have no solutions or only zero solutions and apply these results directly to deriving Diophantine equations with no solutions.

We shall first illustrate the principal method by examining the special equation, with x, y, z not all zero,

$$x^4 + y^4 - 3z^4 = 0. \tag{3}$$

Taking first the case in which 5 is not a divisor of x, y or z, we have the congruence

$$x^4 + y^4 - 3z^4 \equiv 0 (\text{mod } 5), \tag{4}$$

which by Fermat's minor theorem yields the impossibility $1 + 1 - 3 = -1 \equiv 0 \pmod 5$. If, however, 5 divides x, y or z, and since xyz are not all 0, we may divide the equation (3) by 5^a, where 5^a is the greatest power of 5 that divides all the numbers x, y and z. This results in an equation of the type

$$x_1^4 + y_1^4 - 3z_1^4 = 0$$

in which not more than one of the numbers x_1, y_1 and z_1 has the factor 5. This is easily shown to be impossible, hence (3) is impossible. We may secure other equations having no non-zero solutions by multiplying (3) by any number relatively prime to 5 and adding any multiples of 5 to the coefficients. Or, again, we might consider the equation

$$x^8 + 2y^8 + 4z^8 - 8u^8 = 0 \tag{5}$$

with x, y, z and u not all $= 0$. As before, division by 17^a, the highest power of 17 common to all four terms, gives an equation of the type (5) in which two, at most, of the terms are multiples of 17. Since $8 = \frac{17 - 1}{2}$ and

$$x_2^8 \equiv -2y_2^8 - 4z_2^8 + 8u_2^8 (\text{mod } 17),$$

it follows that

$$b \equiv -2c - 4d + 8e (\text{mod } 17),$$

where b, c, d and e range over the values 1, -1 and 0 independently except that not more than 2 of them can $= 0$. If $b \neq 0$, then we have the impossibility of ± 1 congruent to the sum, which is < 17 but > -17, of three even numbers, while if $b = 0$, we note in turn, for the cases $e \neq 0$ and $e = 0$, that $|8e| > |-2c| + |-4d|$ and $|-4d| > |-2c|$, so that the congruence is impossible in any event. As before, multiplication of (5) by numbers relatively prime to 17 and adding any multiples of 17 to the coefficients will give new Diophantine equations having no non-zero solutions.

To obtain general results we extend a method due to H. H. Mitchell[1] for attacking the problem of finding the primes p such that

$$x^m + y^m + 1 \equiv 0 (\text{mod } p),$$

with x and y integers, $xy \not\equiv 0 (\text{mod } p)$, and $p = 1 + mc$ where m is a prime. Consider the congruence

$$k + a_1x_1^m + a_2x_2^m + \ldots + a_sx_s^m \equiv 0 (\text{mod } p), \qquad (6)$$
$$a_1a_2 \ldots a_sx_1x_2 \ldots x_s \not\equiv 0 (\text{mod } p),$$

with m not necessarily prime, but with all the letters representing integers. Using the method described in a previous paper,[1] it follows that if α is a primitive c^{th} root of unity we shall have for some r's

$$k + a_1\alpha^{r_1} + a_2\alpha^{r_2} + \ldots + a_s\alpha^{r_s} \equiv 0 (\text{mod } \mathfrak{p}) \qquad (7)$$

where $\mathfrak{p}$ is a prime ideal divisor of p in the algebraic field defined by $e^{2i\pi/c}$. Let $N(\omega)$ be the norm of ω. Then since the norm is rational we have

$$N(k + a_1\alpha^{r_1} + \ldots + a_s\alpha^{r_s}) \equiv 0 (\text{mod } p). \qquad (8)$$

Now

$$|k + a_1\alpha^{r_1} + \ldots + a_s\alpha^{r_s}| \leqq |k| + |a_1\alpha^{r_1}| + \ldots + |a_s\alpha^{r_s}| \leqq |k| + |a_1| + |a_2| + \ldots + |a_s|$$

since the absolute value of α^r is unity; and the same result holds for any of the conjugates of this number, hence

$$N(k + a_1\alpha^{r_1} + \ldots + a_s\alpha^{r_s}) \leqq (|k| + |a_1| + |a_2| + \ldots + |a_s|)^{\varphi(c)}$$

Therefore, unless one of the factors in the left-hand member of (8) is zero, we must have

$$(|k| + |a_1| + |a_2| + \ldots + |a_s|)^{\varphi(c)} \leqq p,$$

but this is independent of m, so we can select m so large that

$$(|k| + |a_1| + |a_2| + \ldots + |a_s|)^{\varphi(c)} < p, \qquad (9)$$

whence the

THEOREM I. *If c is a given integer > 0, and p is a prime such that $p = 1 + mc$ with m an integer, and $(|k| + |a_1| + |a_2| + \ldots + |a_s|)^{\varphi(c)} < p$, and $k + a_1\alpha^{r_1} + a_2\alpha^{r_2} + \ldots + a_s\alpha^{r_s} \neq 0$ for $\alpha = e^{2i\pi/c}$ and $r_1, r_2, \ldots, r_s$ any integers, then $k + a_1x_1^m + a_2x_2^m + \ldots + a_sx_s^m \equiv 0 (\text{mod } p)$ has no solutions $x_1, x_2, \ldots, x_s$ if $x_1x_2 \ldots x_s \not\equiv 0 (\text{mod } p)$ and $a_1 \ldots a_s \not\equiv 0$.*

Now suppose we consider the equation

$$k + a_1y_1^m + a_2y_2^m + \ldots + a_sy_s^m = 0, \qquad (10)$$

and use the prime p as defined in Theorem I with the conditions given on p

and the a's. By Theorem I, (10) has no solutions in integers $y_1, y_2, \ldots, y_s$ prime to p. Hence, if it has any solutions some of the y's are divisible by p. We may choose our notation so that those not divisible by p are $y_1, y_2, \ldots, y_t$; $0 < t < s$. Hence if (10) is satisfied it gives the congruence

$$k + a_1 y_1^m + a_2 y_2^m + \ldots + a_t y_t^m \equiv 0 \pmod{p},$$
$$y_1 y_2 \ldots y_t \not\equiv 0 \pmod{p}. \quad (11)$$

Now apply the criterion of Theorem I to this congruence, and we see that (11) has no solutions prime to p since the condition (9) is satisfied, provided, however, that $k + a_1\alpha_1^{e_1} + a_2\alpha_2^{e_2} + \ldots + a_t\alpha_t^{e_t} \neq 0$, where $e_1, e_2, \ldots, e_t$ are arbitrary integers. This gives the

THEOREM II. *If c is a given integer > 0, and it is possible to find an integer m and a prime p such that $p = 1 + mc$, also integers $k, a_1, a_2, \ldots, a_s$ so that $a_1 a_2 \ldots a_s \not\equiv 0$ and*

$$(|k| + |a_1| + |a_2| + \ldots + |a_s|)^{\varphi(c)} < p$$

and none of the expressions

$$k + a_1\alpha_1 + a_2\alpha_2 + \ldots + a_s\alpha_s \quad (12)$$

is zero where any j of the α's are zero, $0 \leqq j < s$, and the non-zero α's are any roots of $z^c = 1$ then the equation

$$k + a_1 y_1^m + a_2 y_2^m + \ldots + a_s y_s^m = 0$$

is impossible in integers $y_1, y_2, \ldots, y_s$ except in the case when $k = 0$, and then $y_1 = y_2 = \ldots = y_s = 0$.

Now assume that $k = 0$ and c is a prime, say q. We examine the condition (12) and assume that i of the α's differ from zero, $0 < i \leqq s$. Then these α's are qth roots of unity and the primitive roots satisfy

$$\frac{z^q - 1}{z - 1} = z^{q-1} + z^{q-2} + \ldots + 1 = 0. \quad (13)$$

Assume that $s \leqq q - 2$. If α is a primitive root of $z^q = 1$, all the roots are given by $\alpha, \alpha^2, \ldots, \alpha^q$. Now consider the α's which appear in (12). If none equals α^{q-1}, then (12) is not zero since the left-hand member of (13) is irreducible in the rational field. Say that d of them equal α^{q-1} and call these $\alpha_1, \alpha_2, \ldots \alpha_d$ with $0 < d \leqq i$, where i of the α's differ from zero as noted above. Then using

$$\alpha^{q-1} = -\alpha^{q-2} - \alpha^{q-3} - \ldots - 1,$$

the expression (12) becomes

$$-(a_1 + a_2 + \ldots + a_d)(\alpha^{q-2} + \alpha^{q-3} + \ldots + 1) + a_{d+1}\alpha_{d+1} + a_{d+2}\alpha_{d+2} + \ldots + a_s\alpha_s. \quad (14$$

Consider the terms

$$a_{d+1}\alpha_{d+1} + \ldots + a_i\alpha_i.$$

In number they are $< q - 2$ since $s \leqq q - 2$. Hence if we write (14) in the form

$$h_0 + h_1\alpha + h_2\alpha^2 + \ldots + h_{q-2}\alpha^{q-2} \tag{15}$$

in order for it to be zero each h is zero. But unless $(a_1 + a_2 + \ldots + a_d)$ in (14) is zero then there is a power of α appearing in (15) whose coefficient $-(a_1 + a_2 + \ldots + a_d)$ is not zero and which is not in the set

$$\alpha_{d+1}, \ldots, \alpha_i$$

and this is impossible. Hence

$$a_1 + a_2 + \ldots + a_d = 0, \tag{16}$$

and this gives from (14) and (15)

$$a_{d+1}\alpha_{d+1} + \ldots + a_i\alpha_i = 0$$

and this is impossible unless $d = i$. Hence (16) holds for any d of the a's $0 < d \leqq s$. Whence the

THEOREM III. *If c is a prime and m is an integer such that $p = 1 + mc$ with p prime, then*

$$a_1x_1^m + a_2x_2^m + \ldots + a_sx_s^m = 0 \tag{17}$$

has only the solution $x_1 = x_2 = \ldots = x_s = 0$ provided that $s \leqq c - 2$; the sum of no n of the a's is zero, $0 < n \leqq s$; and

$$(|a_1| + |a_2| + \ldots + |a_s|)^{\varphi(c)} < p.$$

The condition that the sum of no n of the a's is zero is necessary since if

$$a_1 + a_2 + \ldots + a_j = 0$$

then (17) is satisfied with

$$x_1 = x_2 = \ldots = x_j = 1$$
$$x_{j+1} = x_{j+2} = \ldots + x_s = 0.$$

For the least non-trivial value of c, viz., $c = 5$, the theorem shows, for example, that

$$x^m + y^m - 3z^m = 0 \tag{18}$$

is impossible for any m such that $p = 1 + 5m$ and $p > 5^4$ unless $x = y = z = 0$. The smallest such p is 631 giving $m = 126$, so that (18) has no non-zero solutions for $m = 126$, and also

$$ax^{126} + by^{126} + cz^{126} = 0$$

is impossible for any a, b and c such that there exists a k satisfying $ka \equiv kb \equiv 1 \pmod{631}$ and $kc \equiv -3 \pmod{631}$.

[1] Vandiver, H. S., these PROCEEDINGS, **30**, 368-370 (1944).

THE TRANSFORMATION OF DYNAMICAL SYSTEMS OF TWO DEGREES OF FREEDOM

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1. Introduction.—Let D and E by dynamical systems of n degrees of freedom which are referred to the same set of generalized coördinates $x^1, \ldots, x^n$. We represent these systems by the Lagrangian equations

$$D: \quad \frac{d^2x^\alpha}{dt^2} + \Gamma^\alpha_{\mu\nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt} = Q^\alpha, \tag{1.1}$$

$$E: \quad \frac{d^2x^\alpha}{d\tau^2} + \Lambda^\alpha_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = R^\alpha. \tag{1.2}$$

It is assumed that the generalized force vectors Q and R depend on the coördinates x alone and that neither vector vanishes in the region under consideration.[1] The kinetic energies of D and E are assumed to have the form

$$\frac{1}{2} g_{\alpha\beta}(x) \frac{dx^\alpha}{dt} \frac{dx^\beta}{dt} \text{ and } \frac{1}{2} h_{\alpha\beta}(x) \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau}.$$

The above quantities $\Gamma^\alpha_{\mu\nu}$ and $\Lambda^\alpha_{\mu\nu}$ are Christoffel symbols derived from the coefficients $g_{\alpha\beta}$ and $h_{\alpha\beta}$ of these forms, respectively.

In a paper previously written[2] we have given necessary and sufficient conditions for all trajectories of E to be trajectories of D under the assumption that these systems involved more than two degrees of freedom; in particular this led to conditions which were both necessary and sufficient for D and E to have identical trajectories.

In this note we derive the formulas (§ 2) which enable us to say that all theorems obtained in the above paper are likewise valid in the case $n = 2$. However, in the exceptional case of two degrees of freedom results can be stated which are preferable to those previously obtained for systems of more than two degrees of freedom. Conditions are given in § 4 for the

identity of all trajectories of D and E for the general case of systems of two degrees of freedom and in § 5 for the identity of the trajectories of D and E when these systems are conservative.

2. The Fundamental Algebraic Relations.—Assuming all trajectories of E to be trajectories of D and that the parameter transformations $t \to \tau$ and $\tau \to t$ are of class C^2, we have

$$\left(\Phi^{\alpha}_{\mu\nu}\frac{dx^{\beta}}{d\tau} - \Phi^{\beta}_{\mu\nu}\frac{dx^{\alpha}}{d\tau}\right)\frac{dx^{\mu}}{d\tau}\frac{dx^{\nu}}{d\tau} = \left(R^{\alpha}\frac{dx^{\beta}}{d\tau} - R^{\beta}\frac{dx^{\alpha}}{d\tau}\right) - \left(Q^{\alpha}\frac{dx^{\beta}}{d\tau} - Q^{\beta}\frac{dx^{\alpha}}{d\tau}\right)\left(\frac{dt}{d\tau}\right)^2, \quad (2.1)$$

where,

$$\Phi^{\alpha}_{\mu\nu} = \Lambda^{\alpha}_{\mu\nu} - \Gamma^{\alpha}_{\mu\nu},$$

along any trajectory of E. See equation (2.3) of the previous paper[2] (hereafter referred to by the initials T.E.D.). Suppose initially that $dx^{\alpha}/d\tau = kQ^{\alpha}$, where k is a constant. Then from (2.1) we obtain

$$k^3(\Phi^{\alpha}_{\mu\nu}Q^{\beta} - \Phi^{\beta}_{\mu\nu}Q^{\alpha})Q^{\mu}Q^{\nu} = k(R^{\alpha}Q^{\beta} - R^{\beta}Q^{\alpha}).$$

Since the constant k is arbitrary these relations imply

$$R^{\alpha}Q^{\beta} - R^{\beta}Q^{\alpha} = 0, \qquad (\Phi^{\alpha}_{\mu\nu}Q^{\mu}Q^{\nu})Q^{\beta} - (\Phi^{\beta}_{\mu\nu}Q^{\mu}Q^{\nu})Q^{\alpha} = 0.$$

Hence it follows that R^{α} and $\Phi^{\alpha}_{\mu\nu}Q^{\mu}Q^{\nu}$ are proportional to the Q^{α}, i.e.,

$$R^{\alpha} = \psi Q^{\alpha}, \qquad \Phi^{\alpha}_{\mu\nu}Q^{\mu}Q^{\nu} = LQ^{\alpha}, \quad (2.2)$$

where the proportionality factors ψ and L may depend on the coördinates x.

In T.E.D. we showed that the quantities $\Phi^{\alpha}_{\mu\nu}$ must be of the form

$$\Phi^{\alpha}_{\mu\nu} = \delta^{\alpha}_{\mu}\phi_{\nu} + \delta^{\alpha}_{\nu}\phi_{\mu} + A_{\mu\nu}Q^{\alpha}; \quad (2.3)$$

we now show that these relations likewise hold for the case $n = 2$. Considering that the Φ's and Q's are known, the quantities ϕ_{ν} and the symmetric quantities $A_{\mu\nu}$ are determined from (2.3) in accordance with the following equations

$$\left.\begin{aligned}
\Phi^{1}_{22} &= A_{22}Q^{1} && \text{(determines } A_{22}\text{)}\\
\Phi^{2}_{11} &= A_{11}Q^{2} && \text{(determines } A_{11}\text{)}\\
\Phi^{1}_{11} &= 2\phi_1 + A_{11}Q^{1} && \text{(determines } \phi_1\text{)}\\
\Phi^{2}_{22} &= 2\phi_2 + A_{22}Q^{2} && \text{(determines } \phi_2\text{)}\\
\Phi^{1}_{12} &= \phi_2 + A_{12}Q^{1} && \text{(determines } A_{12}\text{).}
\end{aligned}\right\} \quad (2.4)$$

In making these determinations it is supposed that neither of the components Q^1 and Q^2 vanishes; this condition can be realized in the neighbor-

hood of any point P by a proper choice of coördinates since we have assumed that the vector Q does not vanish.

In view of (2.4) all equations (2.3) hold with the possible exception of the equation corresponding to $\alpha = 2, \mu = 1, \nu = 2$. Now put

$$\Phi^2_{12} = \phi_1 + A_{12}Q^2 + K; \tag{2.5}$$

the remaining equation (2.3) will then be satisfied if we can show that $K = 0$. But substituting the values of the Φ's from (2.4) and (2.5) into the second set of equations (2.2) and eliminating the factor L between these two equations we are led to the relation $K(Q^1)^2Q^2 = 0$. Hence, $K = 0$ and (2.3) is necessarily satisfied.

On account of (2.3) and the relation $R^\alpha = \psi Q^\alpha$ it is immediately seen by recourse to T.E.D. that the general discussion and all results obtained in the previous paper are valid for systems of two degrees of freedom.

3. Two-Dimensional Riemann Spaces with Corresponding Geodesics.—Let

$$\lambda_{\alpha\beta}\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt} = \text{const.} \tag{3.1}$$

be a first integral of the differential equations of the geodesics of a Riemann space R with metric defined by $g_{\alpha\beta}$. It is well known that the necessary and sufficient conditions for the existence of this integral are that

$$\lambda_{\alpha\beta,\,\gamma} + \lambda_{\beta\gamma,\,\alpha} + \lambda_{\gamma\alpha,\,\beta} = 0, \tag{3.2}$$

where the "comma" denotes covariant differentiation based on the Christoffel symbols $\Gamma^\alpha_{\mu\nu}$ determined by the $g_{\alpha\beta}$. For the two-dimensional case the relations (3.2), when written in full, become

$$\lambda_{11,\,2} + 2\lambda_{12,\,1} = 0, \quad \lambda_{22,\,1} + 2\lambda_{12,\,2} = 0, \quad \lambda_{11,\,1} = \lambda_{22,\,2} = 0. \tag{3.3}$$

Now determine functions μ_1 and μ_2 as solutions of the two equations

$$\left.\begin{aligned} \lambda_{11}\mu_2 - \lambda_{12}\mu_2 &= \lambda_{11,\,2} \\ -\lambda_{12}\mu_2 + \lambda_{22}\mu_1 &= \lambda_{22,\,1} \end{aligned}\right\} \tag{3.4}$$

Assuming $\det. \mid \lambda_{\alpha\beta} \mid \neq 0$, these equations have a unique solution μ_1 and μ_2. Hence

$$\left.\begin{aligned} \lambda_{12,\,1} &= -{}^1/_2\lambda_{11}\mu_2 + {}^1/_2\lambda_{12}\mu_1 \\ \lambda_{12,\,2} &= {}^1/_2\lambda_{12}\mu_2 - {}^1/_2\lambda_{22}\mu_1 \end{aligned}\right\} \tag{3.5}$$

in consequence of the first two equations (3.3). Equations (3.4) and (3.5) can be written in the combined form

$$\lambda_{\alpha\beta,\,\gamma} = \lambda_{\alpha\beta}\mu_\gamma - {}^1/_2\lambda_{\alpha\gamma}\mu_\beta - {}^1/_2\lambda_{\beta\gamma}\mu_\alpha. \tag{3.6}$$

All conditions (3.3) are now seen to be satisfied by the $\lambda_{\alpha\beta,\,\gamma}$ as given by (3.6) which is thus an allowable expression for these quantities.

From (3.6) we can deduce $\mu_\alpha = (\log \lambda/g)_{,\alpha}$ where $\lambda = \text{det.} \mid \lambda_{\alpha\beta} \mid$ and $g = \text{det.} \mid g_{\alpha\beta} \mid$. Hence μ_α is the gradient of a scalar μ. Putting $\mu = \log \nu$, we have $\nu = c\lambda/g$, where c is a constant. Introducing the function ν into (3.6) this equation can be given the form

$$h_{\alpha\beta,\,\gamma} = 2h_{\alpha\beta}\eta_{,\gamma} + h_{\alpha\gamma}\eta_{,\beta} + h_{\beta\gamma}\eta_{,\alpha} \tag{3.7}$$

where

$$h_{\alpha\beta} = \frac{\lambda_{\alpha\beta}}{\nu^2} \text{ and } \eta = -\frac{1}{2}\log \nu.$$

But (3.7) is the condition[3] that $h_{\alpha\beta}$ defines the metric of a Riemann space R' having the same goedesics as R.[4]

Conversely if a Riemann space R', with metric defined by $h_{\alpha\beta}$, has the same geodesics as R, equations of the form (3.7) hold. By retracing our steps we are thus led to the existence of a quadratic first integral (3.1) of the differential equations of the geodesics of R. The result proved can be stated in the following terms: *The totality of Riemann spaces R', whose geodesics are the same as the geodesics of a given Riemann space R, is derivable, as above, from the set of all first integrals (3.1) of the differential equations of the geodesics of R.*

Now let

$$g^{(i)}_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt} = \text{const.}, \qquad (i = 1, \ldots, s), \tag{3.8}$$

be a basis[5] of quadratic first integrals of the differential equations of the geodesics of R. The above italicized result can now be embodied in the following theorem.

THEOREM. *The geodesics of the two-dimensional Riemann space R' will be the same as the geodesics of the two-dimensional Riemann space R if, and only if,*

$$h_{\alpha\beta} = \left(\frac{\text{det.} \mid g_{\mu\nu} \mid}{\text{det.} \mid c_k g^{(k)}_{\mu\nu} \mid}\right)^2 c_i g^{(i)}_{\alpha\beta}$$

where the $g^{(i)}_{\mu\nu}$ are the coefficients in a basis of quadratic first integrals (3.8) of R and the c's are constants.

If R and R' are Riemann spaces in the strict sense the forms $g_{\alpha\beta}\xi^\alpha\xi^\beta$ and $h_{\alpha\beta}\xi^\alpha\xi^\beta$ are positive definite; then the constants c_i must be such that the $h_{\alpha\beta}$ in the above equations are the coefficients of a positive definite quadratic form.

4. General Correspondence Theorem for Dynamical Systems.—The theorem of § 3 combined with the italicized result at the end of § 7 in T.E.D. leads immediately to the following theorem.

The trajectories of the dynamical system E of two degrees of freedom will be the same as the trajectories of the system D of two degrees of freedom if, and only if,

$$h_{\alpha\beta} = \left(\frac{\text{det.} \mid g_{\mu\nu} \mid}{\text{det.} \mid c_k g^{(k)}_{\mu\nu} \mid}\right)^2 c_i g^{(i)}_{\alpha\beta}$$

$$R^\alpha = c^2 \left(\frac{\text{det.} \mid c_k g^{(k)}_{\mu\nu} \mid}{\text{det.} \mid g_{\mu\nu} \mid}\right)^2 Q^\alpha$$

where the $g^{(i)}_{\mu\nu}$ are the coefficients in a basis of quadratic first integrals (3.8) of the differential equations of the geodesics of D and the c's are constants.

5. *Correspondence of Conservative Systems.*—Suppose now that the systems D and E are conservative so that we have $Q^\alpha = -g^{\alpha\sigma} V_{,\sigma}$ and $R^\alpha = -h^{\alpha\sigma} W_{,\sigma}$ where the potentials V and W depend on the coördinates x alone. Also, let

$$\frac{1}{2} g^{(i)}_{\mu\nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt} + V^{(i)} = \text{const.}, \qquad (i = 1, \ldots, s), \qquad (5.1)$$

be a basis[5] of quadratic first integrals of energy type for the system D.

The requirement that the trajectories of D and E are the same necessitates the vanishing of the quantities $A_{\mu\nu}$ in T.E.D. (see § 7 of T.E.D.). Hence the constants $m_i = 0$ in § 8 of T.E.D. and the equations (8.10) and (8.14) of this paper lead to the same expression for the quantities $h_{\alpha\beta}$ as in the italicized theorem in the preceding section although now the $g^{(i)}_{\alpha\beta}$ are the coefficients of the basis of integrals (5.1).

The relation between the R^α and Q^α in the theorem of § 4 is necessarily satisfied when the trajectories of D and E are the same and this relation is now equivalent to the following

$$W_{,\beta} = c^2 \left(\frac{\text{det.} \mid c_k g^{(k)}_{\mu\nu} \mid}{\text{det.} \mid g_{\mu\nu} \mid}\right)^2 h_{\alpha\beta} g^{\alpha\sigma} V_{,\sigma}$$

$$= c^2 c_i g^{(i)}_{\alpha\beta} g^{\alpha\sigma} V_{,\sigma}$$

$$= c^2 c_i V^{(i)}_{,\beta} = [c^2 c_i V^{(i)}]_{,\beta}$$

where use has been made of the differential conditions for (5.1) to be a first integral of the system (1.1). Hence $W = c^2 c_i V^{(i)} + \text{const.}$ Conversely this equation implies the relation between the vectors R and Q appearing in the theorem of § 4.

These results together with the fact that the relations in the theorem of § 4 are necessary and sufficient for the identity of the trajectories of D and E give us the following theorem. *The trajectories of the conservative system E of two degrees of freedom will be the same as the trajectories of the conservative system D of two degrees of freedom if, and only if,*

$$h_{\alpha\beta} = \left(\frac{\det. \mid g_{\mu\nu} \mid}{\det. \mid c_k g^{(k)}_{\mu\nu} \mid} \right)^2 c_i g^{(i)}_{\alpha\beta}$$

$$W = c^2 c_i V^{(i)} + d$$

where the $g^{(i)}_{\mu\nu}$ *and* $V^{(i)}$ *are the coefficients in a basis of quadratic first integrals (5.1) of the system D and the c's and d are constants.*

The quadratic forms $g_{\alpha\beta}\xi^\alpha\xi^\beta$ and $h_{\alpha\beta}\xi^\alpha\xi^\beta$ are positive definite in the dynamical problem; hence it is understood implicitly in the above theorem and in the theorem in § 4 that the constants c_i are to be chosen so that the $h_{\alpha\beta}$ are the coefficients of a positive definite quadratic form. Moreover, since it is assumed that the force vector Q does not vanish, the constant c appearing in each of these theorems must be different from zero.[1]

[1] It can be shown that if one of the vectors Q and R vanishes at a point the other vector must likewise vanish at that point if the trajectories of the two systems are the same.

[2] "On the Transformation of the Equations of Dynamics," to appear in the June issue of the *Journal of Mathematics and Physics* (1946). Referred to in the text by the initials T.E.D.

[3] See Eisenhart, L. P., *Riemannian Geometry*, Princeton, 1926, p. 133.

[4] The fact that $\lambda_{\alpha\beta}/\nu^2$ defines the metric of a Riemann space having the same geodesics as R appears to be known since the result is stated as a problem on p. 335 of Whittaker, E. T., *Analytical Dynamics*, 4th ed., Dover Publications, 1944. However it is desirable to have the equations leading to this result in order to infer the theorem of § 3.

[5] Thomas, T. Y., "The Fundamental Theorem on Quadratic First Integrals," these PROCEEDINGS, **32**, 10–15 (1946).

THE COEFFICIENTS OF SCHLICHT FUNCTIONS, III

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For functions

$$f(z) = z + a_2 z^2 + a_3 z^3 + \ldots + a_n z^n + \ldots$$

which are regular and schlicht in $\mid z \mid < 1$, the determination of the regions of variability of the coefficients is a classical problem—the so-called coefficient problem for schlicht functions. Given the point $(a_2, a_3, \ldots, a_n)$ we say that

$$f(z) = z + b_2 z^2 + b_3 z^3 + \ldots + b_n z^n + \ldots$$

belongs to the point $(a_2, a_3, \ldots, a_n)$ if $f(z)$ is regular and schlicht in $\mid z \mid < 1$ and

$$b_2 = a_2, b_3 = a_3, \ldots, b_n = a_n.$$

Conversely we say that the point $(a_2, a_3, \ldots, a_n)$ belongs to $f(z)$. The nth region of variability V_n lies in a Euclidean space of $2n - 2$ real dimensions and is the set of points $(a_2, a_3, \ldots, a_n)$ each of which belongs to some $f(z)$. Only V_2 has been determined heretofore, and it is the circle $|a_2| \leqslant 2$.

In this note we outline a method which gives V_n for general n. The boundary of V_3 may be expressed by equations involving only elementary functions, but for $n > 3$ the boundary of V_n is quite complicated and cannot be so expressed. Here, therefore, we shall write out the equations of the boundary only in the case $n = 3$.

We shall outline the method, merely stating results without proof. Detailed proofs will be supplied in a longer paper to appear in the near future.

It is first to be remarked that V_n is closed and bounded. Moreover it is connected; in fact, any point $(a_2, a_3, \ldots, a_n)$ of V_n can be connected to the origin $(0, 0, \ldots, 0)$ by a curve $(a_2t, a_3t^2, \ldots, a_nt^{n-1})$, $0 \leqslant t \leqslant 1$, each point of which lies in V_n. It is readily shown that a point $(a_2, a_3, \ldots, a_n)$ is an interior point of V_n if and only if there is a bounded $f(z)$ belonging to $(a_2, a_3, \ldots, a_n)$.

Write $a_\nu = x_\nu + iy_\nu$ $(\nu = 2, 3, \ldots, n)$. Any real function of the $2n - 2$ coördinates $x_2, y_2, \ldots, x_n, y_n$ may be expressed as a function of $a_2, \bar{a}_2, \ldots, \bar{a}_n, \bar{a}_n$, where $\bar{a}_\nu$ denotes the complex conjugate of a_ν. If G is such a function we define

$$G_\nu = \frac{\partial G}{\partial a_\nu} = \frac{1}{2}\left(\frac{\partial G}{\partial x_\nu} - i\frac{\partial G}{\partial y_\nu}\right), \qquad \bar{G}_\nu = \frac{\partial G}{\partial \bar{a}_\nu} = \frac{1}{2}\left(\frac{\partial G}{\partial x_\nu} + i\frac{\partial G}{\partial y_\nu}\right)$$

(provided the derivatives of first order exist). Let $F(a_2, \bar{a}_2, \ldots, a_n, \bar{a}_n)$ be defined throughout some region B_n containing V_n in its interior, and let F satisfy the following three conditions in B_n:

$$\left.\begin{array}{l} F \text{ is real;} \\ F \text{ and its derivatives of first order } F_\nu \text{ are continuous;} \\ \sum_{\nu=2}^{n} |F_\nu|^2 > 0. \end{array}\right\} \qquad (1)$$

Suppose now that a function F of this type has its maximum value in V_n at the point $(a_2, a_3, \ldots, a_n)$. In view of the third condition (1), we see that $(a_2, a_3, \ldots, a_n)$ must be a boundary point of V_n. If $f(z)$ is any function belonging to $(a_2, a_3, \ldots, a_n)$, then f maximizes F within the family of schlicht functions.

Let $f(z)$ belong to the point $(a_2, a_3, \ldots, a_n)$. Let (a, b) be an arbitrary analytic Jordan arc in $|z| < 1$ but not passing through $z = 0$, and let $p(z)$ be regular in a neighborhood of (a, b) and vanish at the points a and b. On pages 123–125 of "The Coefficients of Schlicht Functions, II [*Duke Mathematical Journal*, 12 (1945)] the authors showed that there is a sequence of functions

$$f_\epsilon(z) = \sum_{m=1}^{\infty} a_m(\epsilon)\, z^m, \qquad a_1(\epsilon) = 1,$$

which are regular and schlicht in $|z| < 1$ for $|\epsilon| < \epsilon_0$ and such that as $\epsilon \to 0$,

$$a_m(\epsilon) = a_m + \epsilon \cdot \frac{1}{2\pi i} \int_a^b p(u) \left\{ \left[\sum_{\nu=1}^{m-1} \nu a_\nu \bar{u}^{m-\nu-2} \right] e^{-2i\tau} - \right.$$

$$\left. \left[\sum_{\nu=1}^{m-1} \nu a_\nu u^{-m+\nu-2} + (m-1) a_m u^{-2} - (f'(u))^2 \sum_{\nu=2}^{m} a_m^{(\nu)} f(u)^{-\nu-1} \right] e^{2i\tau} \right\} ds + o(\epsilon).$$

Here $e^{i\tau}$ is the unit tangent vector of the arc (a, b) and ds is an element of art length.

Since f maximizes F, it may be shown that f satisfies the following differential equation:

$$(zf'(z))^2 \sum_{\nu=2}^{n} A_\nu f(z)^{-\nu-1} = B + \sum_{\nu=1}^{n-1} (B_\nu z^{-n+\nu} + \bar{B}_\nu z^{n-\nu}) \qquad (2)$$

where

$$A_\nu = \sum_{k=\nu}^{n} a_k^{(\nu)} F_k, \qquad B_\nu = \sum_{k=1}^{\nu} k a_k F_{n+k-\nu}, \qquad B = \sum_{k=2}^{n} (k-1) a_k F_k. \quad (3)$$

Here $a_k^{(\nu)}$ are the coefficients of

$$f(z)^\nu = \sum_{k=\nu}^{\infty} a_k^{(\nu)} z^k,$$

B is real, and the right side of (2) is non-negative on $|z| = 1$ and vanishes for at least one point on $|z| = 1$

It may be proved without difficulty that, corresponding to each point P of a set which is everywhere dense in the boundary of V_n, there is at least one function F satisfying conditions (1) which has an absolute maximum in V_n at P. Hence, every function belonging to a point P of this set satisfies a differential equation of type (2). But normalized schlicht functions which satisfy an equation of type (2) form a closed set, and so to every point on the boundary of V_n there belongs at least one function f which satisfies an equation of type (2).

Now let D denote any differential equation of the form (2) where $A_2, A_3, \ldots, A_n, B_1, B_2, \ldots, B_{n-1}$ and B are constants such that B is real and

$$B + \sum_{\nu=1}^{n-1} (B_\nu z^{-n+\nu} + \bar{B}_\nu z^{n-\nu}) \geqslant 0$$

on $|z| = 1$, equality being attained for at least one point on $|z| = 1$. Moreover let D be normalized by the condition that $\sum_{\nu=2}^{n} |A_\nu|^2 = 1$. A

function $f(z)$ will be called a D-function if f is regular in $|z| < 1$ and satisfies some D there, and if f is normalized by the conditions $f(0) = 0, f'(0) = 1$.

One of our main results is the following: There is a one-to-one correspondence between boundary points of V_n and D-functions. In particular, any D-function $f(z)$ is schlicht; and $w = f(z)$ maps $|z| < 1$ onto the w-plane minus a portion containing $w = \infty$ of the piecewise analytic locus which satisfies the Schiffer differential equation

$$\left(\frac{dw}{dt}\right)^2 \cdot \frac{P(w)}{w^{n+1}} = -1 \tag{4}$$

where t is real and $P(w) = A_2 w^{n-2} + A_3 w^{n-3} + \ldots + A_n$.

Let

$$F = Re\{c_2 a_2 + c_3 a_3 + \ldots + c_n a_n\} = \frac{1}{2}\{c_2 a_2 + \bar{c}_2 \bar{a}_2 + \ldots + c_n a_n + \bar{c}_n \bar{a}_n\} \tag{5}$$

where $c_2, c_3, \ldots, c_n$ are any given complex numbers, and let the maximum value of F in V_n be M. Then V_n lies entirely on one side of the $2n - 3$ dimensional hyperplane $F = M$. Since V_n is clearly non-convex for $n > 2$, such supporting hyperplanes can touch V_n only at points which belong to a well-defined subset of the boundary of V_n. The function $w = f(z)$ belonging to any point of this subset of the boundary maps $|z| < 1$ onto the w-plane minus a piece of the locus (4) on which $P(w)$ does not vanish unless the locus lies on a straight line through $w = 0$. Since the boundary slits in the w-plane can have forks in the finite part of the plane only at the zeros of $P(w)$, we see that $f(z)$ maps $|z| < 1$ onto the w-plane minus one or more analytic slits meeting at $w = \infty$, and each of these slits is unforked.

If a function $f(z)$ belonging to a boundary point of V_n satisfies more than one differential equation D, then the function $f(z)$ is algebraic. This follows by dividing one D-equation by the other, the terms $(f'(z))^2$ cancel leaving an algebraic equation in z and $f(z)$.

The boundary of V_n is found by prescribing conditions, involving the coefficients $a_2, a_3, \ldots, a_n$, which insure that the solutions of equations D are regular in $|z| < 1$. It is, of course, necessary to make the zeros of the right side of (2) which lie in $|z| < 1$ correspond to zeros of $P(w)$. We state results for the case $n = 3$.

Let $V_3^{(o)}$ be the subset of V_3 for which a_2 is real; then V_3 is the set of points $(a_2 e^{i\theta}, a_3 e^{2i\theta})$, $0 \leqslant \theta < \pi$, for which (a_2, a_3) belongs to $V_3^{(o)}$. That is, V_3 is obtained from $V_3^{(o)}$ by rotations. The region $V_3^{(o)}$ is symmetrical with respect to the plane $a_2 = 0$, so it is sufficient to describe the part of the boundary of $V_3^{(o)}$ which lies in the half-space $a_2 \geqslant 0$. This portion of

the boundary of $V_3^{(0)}$ is composed of the following two analytic surfaces plus their intersection:

(i) Suppose that $0 \leqslant \alpha \leqslant \frac{\pi}{2}$. Let $-2(\sin\alpha - \alpha\cos\alpha) \leqslant \mu \leqslant 2(\sin\alpha - \alpha\cos\alpha)$ and define $\lambda = 2\cos\alpha\{\log(\cos\alpha) - 1\}$. Then one of the two analytic surfaces is defined by

$$\left.\begin{aligned} a_2 &= \sqrt{\lambda^2 + \mu^2} \\ a_3 &= \lambda^2 + \mu^2 + 2\cos\alpha(\lambda - i\mu) + (2\cos^2\alpha + 1)\frac{\lambda - i\mu}{\lambda + i\mu} \end{aligned}\right\} \quad (6)$$

Any function $w = f(z)$ belonging to a point of this surface generally maps $|z| < 1$ on the w-plane minus a forked slit composed of a ray amp. $(w) =$ constant extending from $w = \infty$ to some finite point where there is a fork composed of two prongs which form angles of $2\pi/3$ with the ray. In special cases one or both prongs may degenerate to a point. If $\alpha = \pi/2$, $f(z)$ maps $|z| < 1$ on the w-plane minus two rays amp. $(w) =$ constant which make an angle of π at $w = \infty$; in this case a_2, a_3 lie on the parabola $a_2 = \mu$, $a_3 = \mu^2 - 1$, $0 \leqslant \mu \leqslant 2$.

(ii) Suppose that $0 < r < 1$, $-\frac{\pi}{2} \leqslant \varphi \leqslant \frac{\pi}{2}$. Let

$$\rho^2 = 1 + 6r^2 + r^4 + 4r(1 + r^2)\cos 2\varphi,$$

$$\tan\alpha = \frac{2r\sin 2\varphi}{1 + 2r\cos 2\varphi + r^2} \qquad \left(-\frac{\pi}{2} < \alpha < \frac{\pi}{2}\right)$$

$$C_1 = \frac{(1 + r)^2}{2r}\cos\varphi, \qquad C_2 = \frac{(1 - r)^2}{2r}\sin\varphi$$

$$\lambda = C_1 \log\frac{\rho}{(1 + r)^2} + C_2\alpha - 2\cos\varphi, \qquad \mu = C_2 \log\frac{\rho}{(1 - r)^2} - C_1\alpha + 2\sin\varphi$$

Then the portion of the boundary of $V_3^{(0)}$ in the half-space $a_2 \geqslant 0$ is completed by the surface defined by the equations

$$\left.\begin{aligned} a_2 &= \sqrt{\lambda^2 + \mu^2} \\ a_3 &= \lambda^2 + \mu^2 + (C_1 + iC_2)(\lambda - i\mu) + \left(r + \frac{1}{r} + \cos 2\varphi\right)\frac{\lambda - i\mu}{\lambda + i\mu} \end{aligned}\right\} \quad (7)$$

If $f(z)$ belongs to a point on this boundary surface of $V_3^{(0)}$ then $w = f(z)$ maps $|z| < 1$ on the w-plane minus a single curved analytic slit extending from $w = \infty$ to some finite point. As $r \to 0$, the slit tends to a straight line amp. $(w) =$ constant and the corresponding f tends to the Koebe function $w = z/(1 + e^{-i\varphi}z)^2$. If $r \to 1$, then $C_1 \to 2\cos\varphi$, $C_2 \to 0$, and

$$\lambda \to 2 \cos \varphi\{\log (\cos \varphi) - 1\}, \qquad \mu \to 2(\sin \varphi - \varphi \cos \varphi).$$

That is, $r = 1$ corresponds to the edge of intersection of the two surfaces. For functions $w = f(z)$ belonging to this edge the polynomial $P(w)$ vanishes on the slit in the w-plane, but one of the two prongs of the fork is absent.